Beer's Law and Thermal Radiative Transfer

Topics:
1. Extinction coefficients and optical depth
2. Beer's Law
3. Radiative transfer equation
4. Thermal RT solutions

Reading: Liou section 1.4; 4.2.2; Thomas & Stamnes 2.7-2.8; 5.4

Extinction Coefficients

The amount of attenuation depends on properties of the medium and distance or amount of material traveled.
Optical path is a product of amount of material and an extinction coefficient:

$$\tau(s) = \int \beta ds \; = \int k \rho_a ds$$

Optical path has no units.

Material		Extinction Coefficient	
s	path length (km)	β	volume extinction (km^{-1})
u	mass/area (g/cm^2)	k	mass extinction (cm^2/g)
Nds	molecules/area (cm^{-2})	σ	molecular cross section (cm^2)

Absorption and scattering are also measured with coefficients: $k_{ext} = k_{sca} + k_{abs}$ where k_{sca} is the scattering coefficient, and k_{abs} is the absorption coefficient.

Optical Depth

Optical depth is optical path in vertical from top down:

$$\tau_\lambda(z) = \int_z^\infty k_\lambda(z')\rho_a(z')dz'$$

For a horizontally homogeneous atmosphere the optical path in a downward direction and optical depth are related by $\tau_\lambda(s) = \tau_\lambda(z)/|\mu|$, where μ is the cosine of the downward zenith angle of path s.

Absorber Amount

If the mass extinction coefficient k is uniform, then the optical depth is

$$\tau_\lambda = k_\lambda u$$

where u is the *absorber amount* between heights z_1 and z_2 (e.g. g/cm^2)

$$u = \int_{z_1}^{z_2} \rho_a(z) dz$$

Using the hydrostatic relation, absorber amount is related to mass mixing ratio of a gas q_a by

$$u = \frac{1}{g} \int_{p_1}^{p_2} q_a dp'$$

For well mixed gases (q_a constant) the absorber amount is proportional to pressure difference across layer.
For more than one absorbing gas, the optical paths add

$$\tau_\lambda = \sum_i k_{i,\lambda} u_i$$

Extinction Law

For no internal sources of radiation:
For an infinitesimal distance, the reduction in radiance is proportional to the incident radiance:

$$dI_\lambda = -k_\lambda \rho_a I_\lambda ds \; = -I_\lambda d\tau_\lambda$$

k_λ is the mass extinction coefficient,
ρ_a is the density of attenuating matter $(\mathrm{g/cm^3})$,
ds is the infinitesimal distance, and
$d\tau_\lambda$ is the infinitesimal optical path at wavelength λ.

Beer-Bouguer-Lambert Law

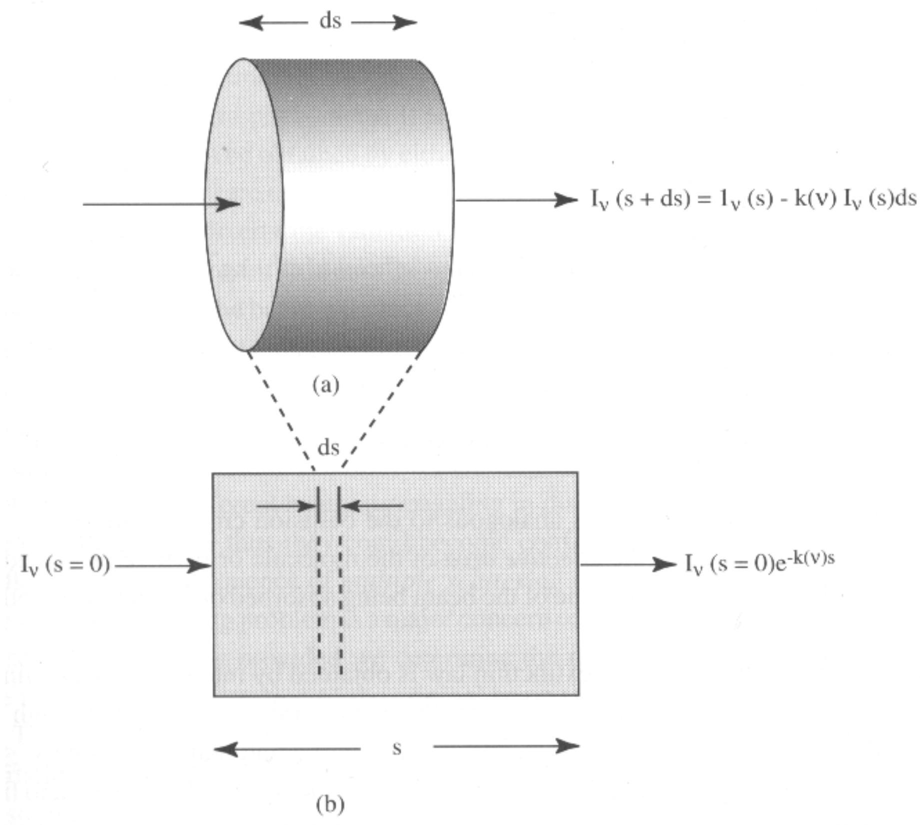
Integrate the monochromatic radiative transfer equation for no sources:

$$\int_0^s \frac{dI_\lambda}{I_\lambda} = -\int_0^s k_\lambda \rho_a ds' \; = -\int_0^{\tau_\lambda} d\tau'_\lambda$$

Solution is exponential decay of radiance:

$$I_\lambda(s) = I_\lambda(0) \; \exp\left(-\int_0^s k_\lambda \rho_a ds'\right) \; = \; I_\lambda(0)\, e^{-\tau_\lambda(s)}$$

where $\tau_\lambda(s)$ is the optical path along ray s.



(a) Intensity passing through a thin slab suffers extinction proportional to the path length ds.
(b) Intensity passing through a finite path length s suffers exponential extinction. [Thomas & Stamnes, Fig. 2.4]

Transmissivity and Absorptivity

The monochromatic *transmissivity* is defined as the fraction of radiance transmitted through a layer

$$\mathcal{T}_\lambda = \frac{I_\lambda(s)}{I_\lambda(0)} = e^{-\tau_\lambda/\mu}$$

By energy conservation the fraction transmitted, absorbed, and reflected equals 1:

$$\mathcal{T} + A + R = 1$$

For a nonscattering layer, the absorptivity is $A = 1 - \mathcal{T}$.

Transmissivity Example

CO_2 has a volume mixing ratio of about 360 ppm in the Earth's atmosphere. Calculate the absorber amount for a layer from 995 to 1005 mb.
Convert from volume mixing ratio to mass mixing ratio using molecular masses of CO_2 and air [360 ppmv(44.0/29.0) = 546 ppmm].

$$u = \frac{q_a}{g}\Delta p \; = \frac{(5.46 \times 10^{-4})(100 \text{ N m}^{-2}/\text{mb})(10 \text{ mb})}{9.8 \text{ m/s}^2} = 0.056 \text{ kg/m}^2$$

The mass absorption coefficient for CO_2 at 1000 mb pressure and 296 K temperature at a wavenumber of 700.39 cm^{-1} is $16.3 \text{ m}^2/\text{kg}$. Calculate the optical depth and transmissivity for $\mu = 0.6$.

$$\tau_\nu = k_\nu u \; = (16.3 \text{ m}^2/\text{kg})(0.056 \text{ kg/m}^2) = 0.91$$

$$\mathcal{T}_\nu = e^{-\tau_\nu/\mu} \; = e^{-0.91/0.6} = 0.218$$

Langley Plots

The transmission of flux in the collimated beam of sunlight is described by Beer's Law for monochromatic radiance in a plane-parallel atmosphere:

$$F_\lambda = F_{0,\lambda} \exp(-\tau_\lambda/\mu_0)$$

τ_λ is the optical depth of the atmosphere,
μ_0 is the cosine of the solar zenith angle.

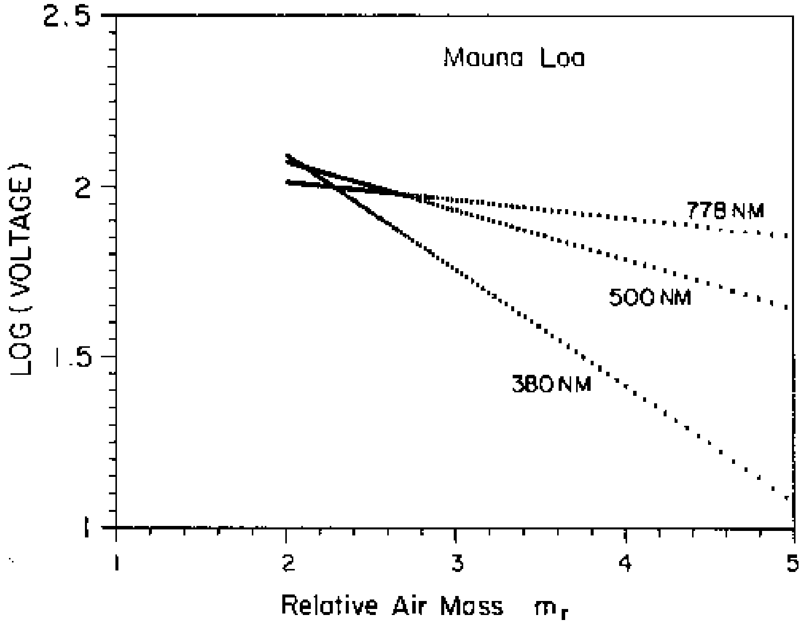
A Langley plot is obtained from a sun photometer as the sun rises.
It is a plot of $\ln F_\lambda$ vs. air mass $m_r = 1/\mu_0$:

$$\ln F_\lambda \;=\; \ln F_{0,\lambda} \;-\; \tau_\lambda \,(1/\mu_0)$$

The slope is the optical depth.
The Sun photometer does not have to be calibrated, just linear so $V = aF$.

$$\ln V = \ln V_0 \;-\; \tau \,(1/\mu_0)$$

Usually can ignore curvature of atmosphere (so plane-parallel) for $\theta < 80°$.



An example of a Langley plot for three wavelengths showing Sun photometer log voltage vs. air mass ($\sec\theta_0$) for clear conditions. [Stephens, 1994, Fig. 6.1]

Radiative Transfer Equation with Sources

There are two sources of radiation we will consider:
1) thermal emission,
2) scattering from other directions into beam.

The sources of radiation increase the radiance along direction Ω by
$dI_\lambda(s,\Omega) = J(s,\Omega)\beta ds$
which defines the *source function* $J(s,\Omega)$ with units of radiance.
Including extinction the radiative transfer equation then may be written:

$$\frac{dI_\lambda(s,\Omega)}{ds} = \beta\left[-I_\lambda(s,\Omega) + J_\lambda(s,\Omega)\right]$$

where $I(s,\Omega)$ is the radiance at point s along path in direction Ω, and β is the volume extinction coefficient. The extinction term (first on right side) decreases the radiance along a ray, while the source term increases the radiance.
The RTE can be expressed with a gradient operator

$$\Omega\cdot\nabla I_\lambda(\mathbf{x},\Omega) = \beta\left[-I(\mathbf{x},\Omega) + J(\mathbf{x},\Omega)\right]$$

where $\mathbf{x}$ is the space coordinate. $\Omega\cdot\nabla I_\lambda$ is called the *streaming term*.

Plane-Parallel Radiative Transfer Equation

Usually we assume atmospheres are horizontally uniform, so only the vertical derivative remains in the RTE:

$$\mu \frac{dI_\lambda(z,\Omega)}{dz} = \beta\left[-I_\lambda(z,\Omega) + J_\lambda(z,\Omega)\right]$$

Often use optical depth as vertical coordinate, $d\tau = -\beta dz$.
This simplifies RTE to

$$\mu \frac{dI_\lambda(\tau,\mu,\phi)}{d\tau} = I_\lambda(\tau,\mu,\phi) - J_\lambda(\tau,\mu,\phi)$$

The sign changed because optical depth increases downward.

Radiative Transfer Equation for Thermal Emission

If there is no scattering then the absorptivity of path ds is $a_\lambda = \beta_\lambda ds$.
In LTE Kirchhoff's Law says emissivity equals absorptivity: $\epsilon_\lambda = \beta_\lambda ds$.
In LTE, thermal emission from path ds is $dI_\lambda = \epsilon B_\lambda(T) = B_\lambda(T)\beta_\lambda ds$.
Therefore, the source function equals the Planck function, $J = B_\lambda(T)$.

Plane-parallel thermal RTE has no dependence on azimuth angle ϕ because thermal emission is isotropic and boundary conditions are azimuthally symmetric.

$$\mu \frac{dI_\lambda(\tau,\mu)}{d\tau} = I_\lambda(\tau,\mu) - B_\lambda[T(\tau)]$$

Solutions for Thermal Radiative Transfer

Plane-parallel thermal emission radiative transfer equation (no scattering):

$$\mu \frac{dI_\lambda(\tau,\mu)}{d\tau} = I_\lambda(\tau,\mu) - B_\lambda[T(\tau)]$$

This is solved with integrating factor $\frac{1}{\mu}e^{-\tau/\mu}$:

$$e^{-\tau/\mu}\frac{dI(\tau,\mu)}{d\tau} - \frac{1}{\mu}e^{-\tau/\mu}I(\tau,\mu) \; = -\frac{1}{\mu}e^{-\tau/\mu}B(\tau)$$

Use chain rule of calculus:

$$\frac{d}{d\tau}\left(e^{-\tau/\mu}I(\tau,\mu)\right) \; = -\frac{1}{\mu}e^{-\tau/\mu}B(\tau)$$

Boundary conditions: blackbody surface (at $\tau = \tau_*$) and none incident at TOA:

$$I(\tau_*, \mu > 0) = B[T(\tau_*)] \qquad I(0, \mu < 0) = 0$$

Integrate equation for upwelling radiance:

$$\left[e^{-\tau/\mu} I^+(\tau, \mu)\right]_{\tau}^{\tau_*} \; = - \int_{\tau}^{\tau_*} e^{-\tau'/\mu} B(\tau') d\tau'/\mu$$

Solution for upwelling radiance at optical depth τ is

$$I^+(\tau, \mu) \; = e^{-(\tau_* - \tau)/\mu} I(\tau_*, \mu) + \int_{\tau}^{\tau_*} e^{-(\tau' - \tau)/\mu} B(\tau') d\tau'/\mu$$

$\tau' - \tau$ is optical depth from observer down to integrating point.

The top of atmosphere radiance is

$$I^+(\mu, 0) = e^{-\tau^*/\mu} B[T(\tau^*)] + \int_0^{\tau^*} B[T(\tau')] \exp(-\tau'/\mu) \frac{d\tau'}{\mu}$$

First term is surface contribution; integral is atmospheric contribution. For each differential layer, $d\tau'/\mu$ is emissivity and $\exp(-\tau'/\mu)$ is transmission to top.

Thermal RT for Single Isothermal Layer

Consider layer from τ_1 to τ_2; optical thickness $\Delta\tau$.

Upwelling radiance at top of layer is

$$I^+(\tau_1, \mu) \; = e^{-(\tau_2 - \tau_1)/\mu} I(\tau_2, \mu) + \int_{\tau_1}^{\tau_2} e^{-(\tau' - \tau_1)/\mu} B(\tau') d\tau'/\mu$$

Isothermal layer (at T) implies Planck function is constant:

$$I^+(\tau_1, \mu) \; = e^{-\Delta\tau/\mu} I(\tau_2, \mu) + B(T) \int_0^{\Delta\tau} e^{-\tau'/\mu} d\tau'/\mu$$

Solution is

$$I^+(\tau_1, \mu) \; = e^{-\Delta\tau/\mu} I(\tau_2, \mu) + B(T)[1 - e^{-\Delta\tau/\mu}]$$

Transmissivity of layer is $\mathcal{T} = e^{-\Delta\tau/\mu}$; emissivity is $\epsilon = 1 - e^{-\Delta\tau/\mu}$.

This solution also applies for an optically thin atmosphere: then T is the absorber weighted atmospheric temperature.

Single Layer Example

At a wavelength of 10.14 μm the optical depth of a standard midlatitude summer atmosphere is 0.27. Most absorption in the Earth's atmospheric window is due to water vapor near the surface. Assume the atmosphere emits as a layer at 285 K. The surface is a blackbody with temperature of 295 K.

Calculate the upwelling radiance at $\theta = 0°$ and $\theta = 60°$.

The two Planck radiances needed are:

$$B_\lambda(295) = 9.12 \text{ W m}^{-2}\text{sr}^{-1}\mu\text{m}^{-1} \quad B_\lambda(285) = 7.69 \text{ W m}^{-2}\text{sr}^{-1}\mu\text{m}^{-1}$$

The transmissions are:

$$\mathcal{T}(0°) = \exp(-\tau/\cos\theta) = \exp(-0.27/1.0) = 0.763. \quad \mathcal{T}(60°) = 0.583.$$

The TOA radiance is: $I_\lambda = \mathcal{T} B_\lambda(T_s) + (1 - \mathcal{T})B_\lambda(T_a)$

$$I_\lambda(0°) = 0.763(9.12) + 0.237(7.69) = 8.78 \text{ W m}^{-2}\text{sr}^{-1}\mu\text{m}^{-1} \quad T_b = 292.7\ K$$

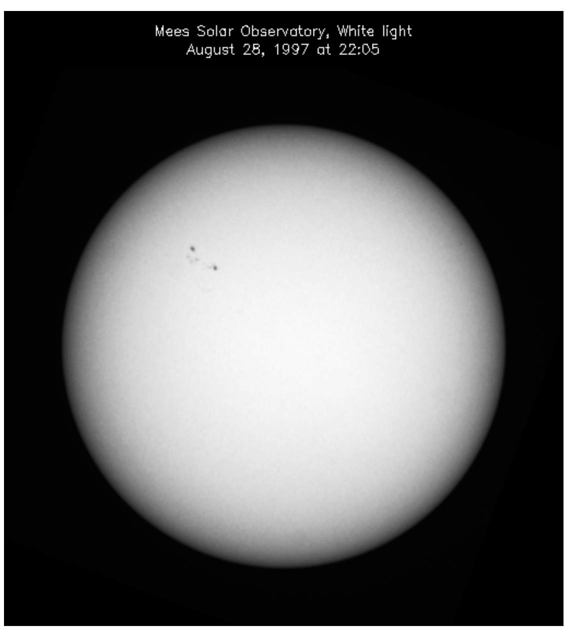
$$I_\lambda(60°) = 0.583(9.12) + 0.417(7.69) = 8.53 \text{ W m}^{-2}\text{sr}^{-1}\mu\text{m}^{-1} \quad T_b = 291.0\ K$$

Note that the brightness temperatures are between T_s and T_a because the upwelling radiance is a weighted average of two Planck blackbody radiances.

Downwelling radiance at surface: $I_\lambda = (1 - \mathcal{T})B_\lambda(T_a)$

$$I_\lambda(0°) = 0.237(7.69) = 1.82 \text{ W m}^{-2} \text{ sr}^{-1}\ \mu\text{m}^{-1}$$

This example shows the decrease in radiance for more oblique viewing angles. This *limb darkening* is due to more of the emission for a slant path originating from the colder atmosphere, or, in general, from higher colder layers in an atmosphere.



A photograph of the solar disk in white light showing limb darkening towards the edges.

Discrete Multilayer Thermal RT Solution

Computer algorithms can integrate the thermal RTE by iterating the single layer solution. This assumes the layers are thin enough so temperatures are close to the mean layer temperature.

For upwelling radiance:

Start at blackbody surface $I_{N+1}(\mu) = B(T_s)$

Iterate going upward (i decreasing):

$$I_i(\mu) \ = e^{-\Delta\tau_i/\mu} I_{i+1}(\mu) + B(T_i)[1 - e^{-\Delta\tau_i/\mu}]$$

where $\Delta\tau_i$ is optical depth of i'th layer, T_i is mean layer temperature.

For downwelling radiance:

Start at TOA $I_1(\mu) = 0$

Iterate going downward (i increasing):

$$I_{i+1}(\mu) \ = e^{-\Delta\tau_i/\mu} I_i(\mu) + B(T_i)[1 - e^{-\Delta\tau_i/\mu}]$$

The radiance exitting one computational layer is the transmitted radiance plus the emitted radiance.

This equation is used to propagate a ray at angle μ through many layers.

Flux Radiative Transfer

Hemispheric flux is obtained from an integral over directions:

$$F^{\uparrow} = 2\pi \int_0^1 \mu \left\{ e^{-\tau^*/\mu} B[T(\tau^*)] + \int_0^{\tau^*} B[T(\tau')] \exp(-\tau'/\mu) \frac{d\tau'}{\mu} \right\} d\mu$$

The integration may be expressed with exponential integral functions.

In practice, the angular integration is performed with quadrature:

$$F^{\uparrow} = 2\pi \sum_{j=1}^{n} w_j \mu_j I(\mu_j)$$

Double-Gauss quadrature integrates polynomials in μ exactly.

Double-Gauss quadrature sets:

$n = 2$:	$\mu_1 = 0.211325$	$w_1 = 0.500000$	$\mu_2 = 0.788675$	$w_2 = 0.500000$
$n = 4$:	$\mu_1 = 0.069432$	$w_1 = 0.173927$	$\mu_2 = 0.330009$	$w_2 = 0.326073$
	$\mu_3 = 0.669991$	$w_3 = 0.326073$	$\mu_4 = 0.930568$	$w_4 = 0.173927$