Multi-Stream and 3D Radiative Transfer

Topics:
1. Eigen-matrix method
2. Nakajima-Tanaka method for radiances
3. Plane-parallel radiance results (see lab 9)
4. 3D radiative transfer methods
5. 3D radiative transfer phenomena

Reading: Thomas 8.5-8.9

Eigen-matrix Method

The eigen-matrix or "discrete ordinate" method is the most standard numerical technique for plane-parallel radiative transfer, e.g. DISORT.

The eigen-matrix method:

1. Efficiently solves the discrete ordinate matrix RTE.
2. The mathematical solution is in terms of eigenvalues (k_j, $e^{\pm k_j \tau}$) and eigenvectors $G_j(\pm\mu_i)$ of the discrete RTE.
3. Special care must be taken to avoid numerical ill-conditioning.
4. Gives exactly same results as doubling-adding (same matrix equations).

Rearrange matrix form of RTE:

$$\frac{d}{d\tau}\begin{pmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{pmatrix} = \begin{pmatrix} -t & -r \\ r & t \end{pmatrix}\begin{pmatrix} \mathbf{I}^+ \\ \mathbf{I}^- \end{pmatrix} - \begin{pmatrix} \mathcal{J}^+ \\ \mathcal{J}^- \end{pmatrix}$$

where $\mathbf{I}^\pm$ are the upward and downward discrete ordinates radiance vectors $I_m(\mu_j)$ of length N. The t and r matrices are defined by

$$t_{j,j'} = \frac{1}{\mu_j}\left[\frac{\omega}{2} w_{j'} P_m(\mu_j, \mu_{j'}) - 1\right] \qquad r_{j,j'} = \frac{1}{\mu_j}\frac{\omega}{2} w_{j'} P_m(\mu_j, -\mu_{j'})$$

and the solar source vector is

$$\mathcal{J}^\pm = \frac{\pm 1}{\mu_j}\frac{\omega}{4\pi} P_m(\pm\mu_j, -\mu_0) S_0 e^{-\tau/\mu_0}$$

μ_j are the quadrature angles, w_j the quadrature weights, and $P_m(\mu, \mu')$ is the Fourier azimuth transformed phase function.

Eigenvalues and Eigenvectors

Try a solution to the homogeneous equation ($\mathcal{J}^{\pm} = 0$) of the form

$$\mathbf{I}^{\pm} = \mathbf{G}^{\pm} e^{-k\tau}$$

The resulting $2N \times 2N$ matrix equation is

$$\begin{pmatrix} t & r \\ -r & -t \end{pmatrix} \begin{pmatrix} \mathbf{G}^{+} \\ \mathbf{G}^{-} \end{pmatrix} = k \begin{pmatrix} \mathbf{G}^{+} \\ \mathbf{G}^{-} \end{pmatrix}$$

The solutions are eigenvalues k_j and eigenvectors $\mathbf{G}_j^{\pm}$.

Due to symmetries, the matrix equation can be reduced to rank N

$$(t - r)(t + r)(\mathbf{G}^{+} + \mathbf{G}^{-}) = k^2(\mathbf{G}^{+} + \mathbf{G}^{-})$$

Thus the eigenvalues come in N positive/negative pairs ($\pm k_j$).

The General Solution

The solution is a linear combination of the homogeneous solutions,
plus the particular solution ($\mathbf{Z}_0^{\pm} e^{-\tau/\mu_0}$):

$$\mathbf{I}^{\pm} = \sum_{j=1}^{N} C_{-j} \mathbf{G}_{-j}^{\pm} e^{k_j \tau} + \sum_{j=1}^{N} C_j \mathbf{G}_j^{\pm} e^{-k_j \tau} \; + \; \mathbf{Z}_0^{\pm} e^{-\tau/\mu_0}$$

where C_j are the constants of integration.
The particular solution discrete ordinate vector $\mathbf{Z}_0^{\pm}$ is found from by substituting in $\mathbf{Z}_0^{\pm} e^{-\tau/\mu_0}$ and solving the resulting matrix equation.

The $2N$ constants C_j are determined from the boundary conditions by setting the general solution to the incident radiance vector at top and bottom of layer.

The $2N \times 2N$ system of equations is ill-conditioned unless the constants are rescaled, e.g. $C_{-j} = C'_{-j} e^{-k_j \tau^*}$, so have only negative exponents.

Multiple Layers and Outgoing Radiances

Multiple homogeneous layers are solved by matching radiances at all internal boundaries, giving $2N \times L$ constants (L is number of layers).

The system of equations is block tridiagonal, since constants for one layer only couple with adjacent layers.

Radiances are found by integrating the source function for each layer:

$$I_m(0,\mu) = I_m(\tau^*,\mu)e^{-\tau/\mu} + \frac{\omega}{2}\sum_{j=-N}^{N} w_j P_m(\mu,\mu_j)\int_0^{\tau^*} I_m(\tau,\mu_j)\, e^{-\tau/\mu}\, \frac{d\tau}{\mu}$$

The general solution for the discrete ordinate radiance is put in so the integral may be done analytically.
Then the Fourier series in azimuth is summed for the output radiances:

$$I(0,\mu,\phi) = \sum_{m=0}^{M} I_m(0,\mu)\cos m\phi$$

Nakajima-Tanaka Method for Radiances

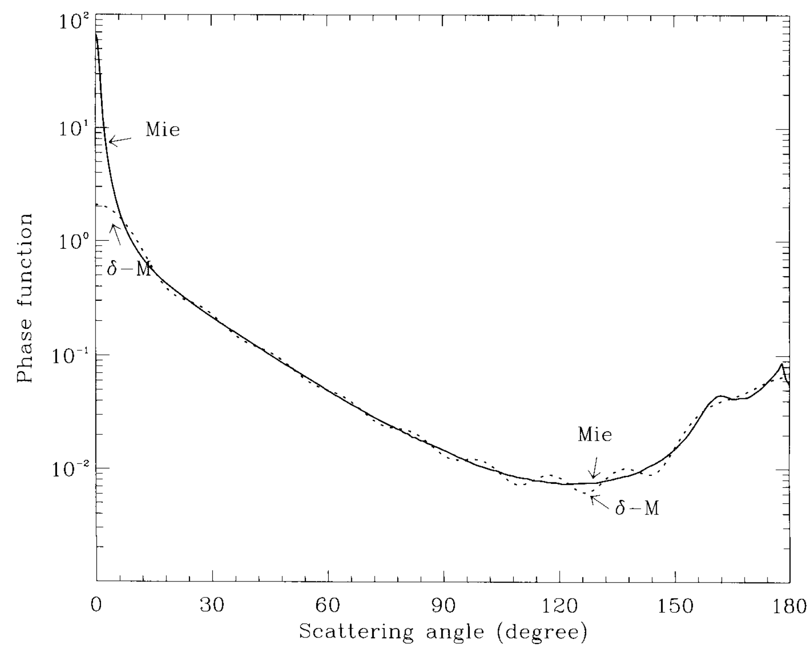
The delta-M approximation is used to scale the RTE and give a smoother phase function. For N “streams” the Legendre phase function coefficients are scaled by

$$\chi_l' = \frac{\chi_l - f}{1-f} \quad l < 2N \quad f = \chi_{2N} \qquad \chi_l = \frac{\omega_l}{2l+1}$$

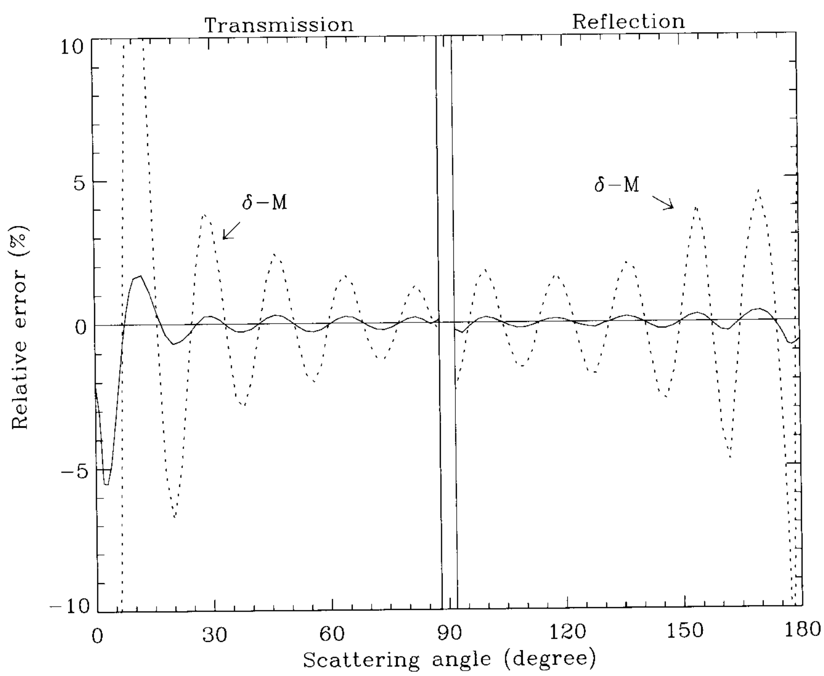
Delta-M gives accurate fluxes from discrete ordinate RTE, but still has oscillations in the radiance field.

The *Nakajima-Tanaka method* uses the full untruncated phase function for the first (and second) order scattered radiance.

The TMS method subtracts off the delta-scaled single scattering contribution, and adds in the exact single scattering solution (with unscaled ω and $P(\Theta)$, scaled τ).



Scattering phase function computed by Mie theory and the δ-M representation for $N = 10$. [Thomas & Stamnes; Fig. 8.5]



Relative error of the reflected and transmitted intensities computed by strict application of δ-M and by applying the Nakajima-Tanaka TMS method (solid line). The TMS correction is the difference between the singly scattered intensity computed from the exact phase function and that from the δ-M-scaled phase function. This example pertains to vertical solar illumination of a homogeneous slab of optical thickness 0.8 with phase function in the previous figure. [Thomas; Fig. 8.6]

3D Radiative Transfer

Clouds are not plane-parallel! Horizontal radiative transport is important and three-dimensional radiative transfer methods are required. The 3D RTE is

$$\sin\theta\cos\phi\frac{\partial I(\mathbf{x},\Omega)}{\partial x}+\sin\theta\sin\phi\frac{\partial I(\mathbf{x},\Omega)}{\partial y}+\cos\theta\frac{\partial I(\mathbf{x},\Omega)}{\partial z}=\beta(\mathbf{x})\ [-I(\mathbf{x},\Omega)+J(\mathbf{x},\Omega)]$$

Two classes of 3D radiative transfer methods:

1. Monte Carlo - simulates photon trajectories.
   Faster for computing a small number of outputs.

2. Explicit methods - solve for whole radiation field (e.g. SHDOM).
   Faster for computing many outputs.

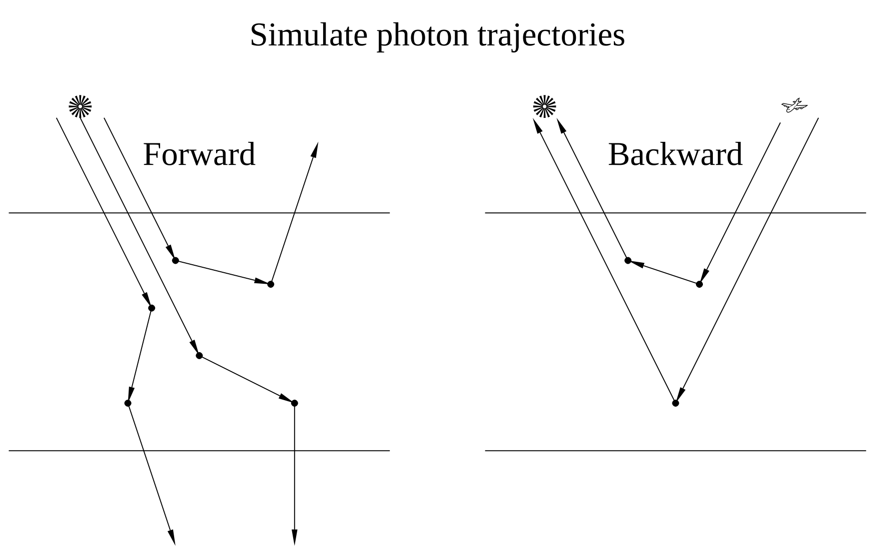
Monte Carlo Radiative Transfer

Usually thought of as a simulation of many random photon trajectories.

Can also be thought of as a Monte Carlo integration of the successive order of scattering equations.

Forward Monte Carlo method: send in photons from sun, absorb and scatter them in medium, and count number exitting in particular locations.

Backward Monte Carlo method: send in photons *backward* from observer, scatter them in medium, at each order of scattering find flux contribution from sun.



Schematic diagram illustrating photon trajectories for forward and backward Monte Carlo RT.

Forward Monte Carlo Method

Send in a photon from Sun at incident direction at random top locations.

For each photon and order of scattering:

- Pick a random transmission $\mathcal{T}$ between 0 and 1.
- Trace photon in Ω direction ($\mathbf{x} = \mathbf{x}_0 + s\Omega$). Accumulate optical path in extinction field until $\int \beta(s)ds = -\ln \mathcal{T}$.
  Check if photon leaves medium, if so add weight to flux counter.

Scattering event at location $\mathbf{x}$:

- Multiply photon weight by single scattering albedo ω.
- Pick a random scattering fraction η and polar angle ζ.
  Get scattering angle Θ from $\eta = \frac{1}{2}\int_0^{\Theta} P(\Theta')d(\cos\Theta')$.
- Compute new direction Ω from old direction and (Θ, ζ).

Continue tracing and scattering photon until weight is almost zero or photon leaves the medium. Then start with next photon.

Backward Monte Carlo

Backward Monte Carlo can be thought of as a successive order of scattering solution with integrals performed with random points (Monte Carlo integration).
Each scattering adds three integrals to order of scattering solution:

$$I_n(\mathbf{x},\Omega) \ = \int_0^{s_{bnd}} J_n(\mathbf{x}-\Omega s,\Omega)\ \exp\left[-\int_0^s \beta(s')ds'\right]\ \beta(s)\ ds$$

$$J_{n+1}(\mathbf{x},\Omega) = \frac{\omega}{4\pi}\int\int_{4\pi} P(\Omega,\Omega')\ I_n(\mathbf{x},\Omega')\ d\Omega'$$

The total radiance is sum of all orders of scattering:

$$I(\mathbf{x},\Omega) = \sum_{n=1} I_n(\mathbf{x},\Omega)$$

The photon trajectories are traced back from the observer location just like forward Monte Carlo. At each scattering event the contribution from the Sun is found by adding in

$$\frac{\omega}{4\pi}P(\Omega,-\Omega_0)S_0 e^{-\tau(\mathbf{x})}$$

where $\tau(\mathbf{x})$ is the optical path towards the Sun from the scattering point $\mathbf{x}$.
It is easy to do backwards Monte Carlo for thermal emission by including emission term in source function.

Monte Carlo Characteristics

Monte Carlo RT methods are very flexible: can do any geometry, etc.

Like successive orders of scattering method, Monte Carlo is relatively fast for optically thin or absorbing atmospheres.

Error in fluxes due to counting statistics: $\sim 1/\sqrt{N}$, where N is number of photons counted for flux.
Roughly 1000000 photons required to get average albedo to 0.001.
Number of photons needed is proportional to number of outputs.

Spherical Harmonic Discrete Ordinate Method (SHDOM)

SHDOM is the fastest explicit 3D atmospheric radiative transfer code.

The source function (not radiance field) is represented to save computer memory:
Spatial discretization: discrete grid points
Angular disretization: spherical harmonic expansion $J_{lm}(\mathbf{x}_i)$
Adaptive spatial grid and spherical harmonic series.

Solution method:

1. Transform source function to discrete ordinates.
2. Integrate RTE source function along discrete ordinates.
3. Transform radiance back to spherical harmonics.
4. Calculate source function from radiance.

Code available at `http:/nit.colorado.edu/`

Features:
- Shortwave and longwave radiation
- Monochromatic or correlated k-distribution broadband
- General optical properties may be specified
- Spatially variable bidirectional surface reflection
- Outputs: radiances, hemispheric fluxes, net fluxes, heating rate.

Finite Cloud Effects

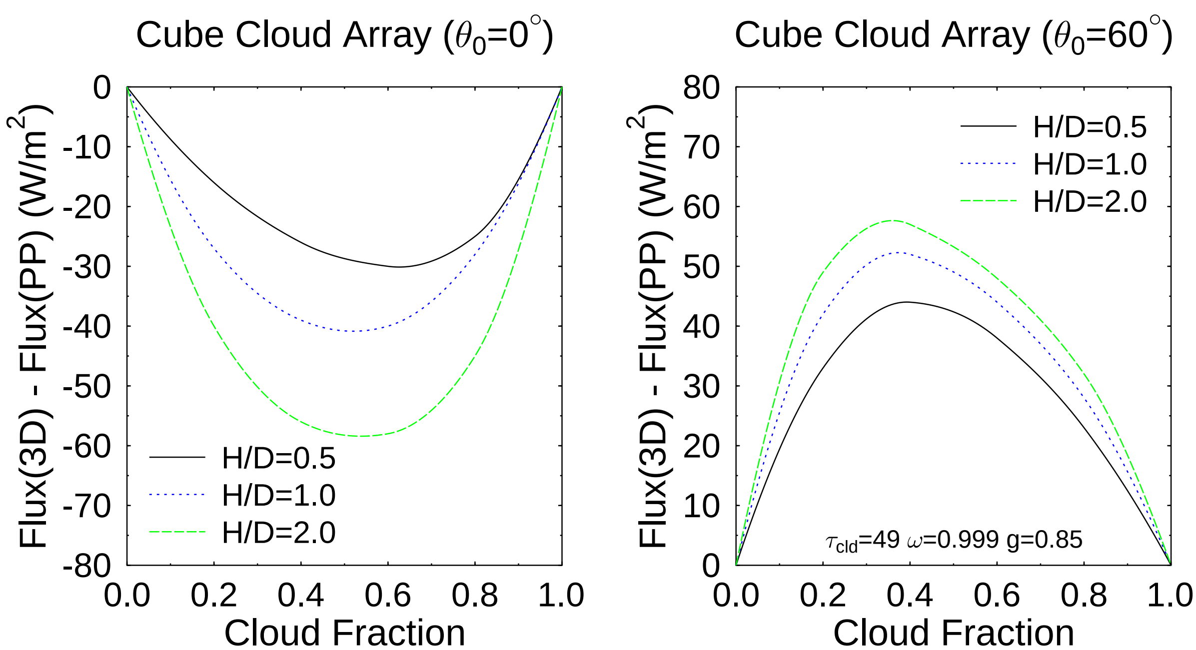
Monte Carlo radiative transfer models first used to study finite (e.g. cube) clouds.

Experiment: how does finite cloud field albedo compare with albedo from plane-parallel RT weighted by cloud fraction?

Results:

High sun (small θ_0): finite clouds have lower albedo than plane-parallel
- photons leak out sides in downward direction.

Low sun (large θ_0): finite clouds have higher albedo than plane-parallel
- extra upward flux from cloud side illumination.



The difference between periodic cubic cloud field and cloud fraction weighted plane-parallel radiative fluxes in W/m^2 as a function of cloud fraction. H/D is the ratio of cloud depth to cloud width. [Welch and Wielicki (1984)]

Overcast Cloud Effects

Monte Carlo modeling in realistic modeled stratocumulus clouds shows that the *Independent Pixel Approximation* is accurate for domain average fluxes.

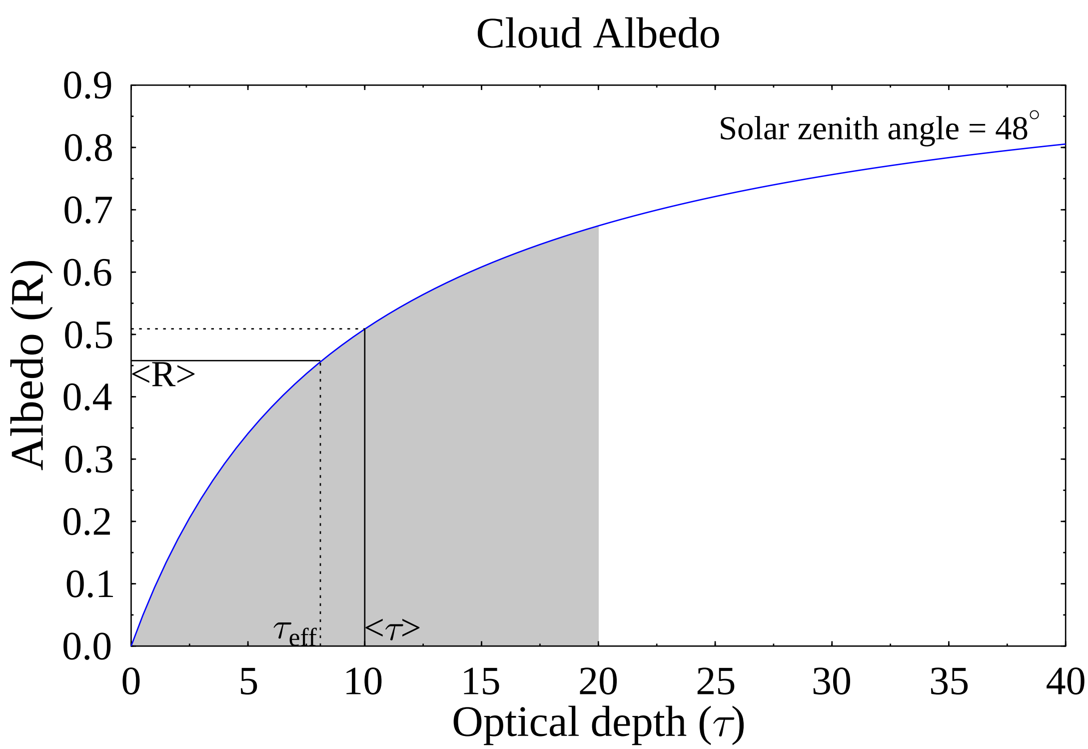
Independent Pixel Approximation: do plane-parallel RT on each column independently, then average results (Cahalan, et al., 1994).

Equivalent to integrating over the optical depth probability distribution $p(\tau)$

$$\langle R \rangle_{ipa} = \int p(\tau) R_{pp}(\tau) d\tau$$

The *plane-parallel albedo bias* is the difference between plane-parallel and IPA albedo. It is due to nonlinear $R(\tau)$ relation, not due to horizontal transport.

Fast methods for doing IPA have been developed for climate models.

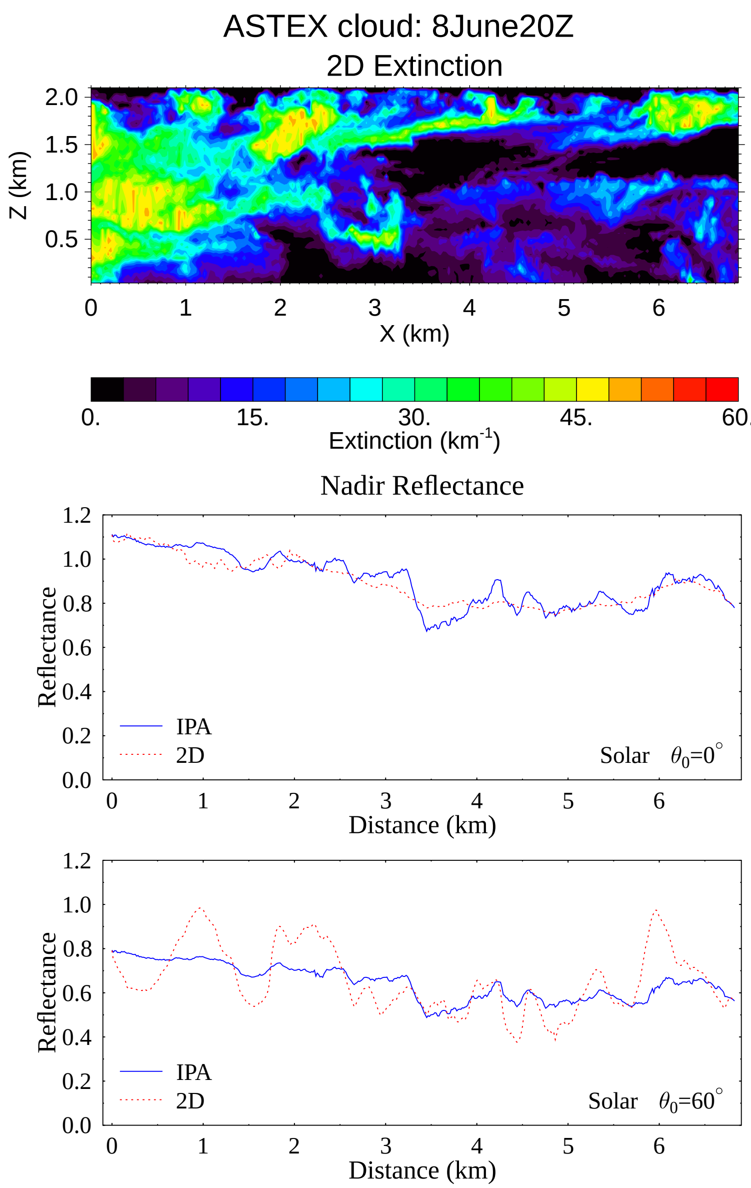


An illustration of the cause of the plane-parallel albedo bias due to optical depth variability. The mean albedo from a uniform optical depth distribution from 0 to 20 is $\langle R \rangle = 0.458$, which is less than the albedo for the mean optical depth ($\langle \tau \rangle = 10$) of $R_{\langle \tau \rangle} = 0.509$.

Overcast Cloud Variability Effects on Radiances

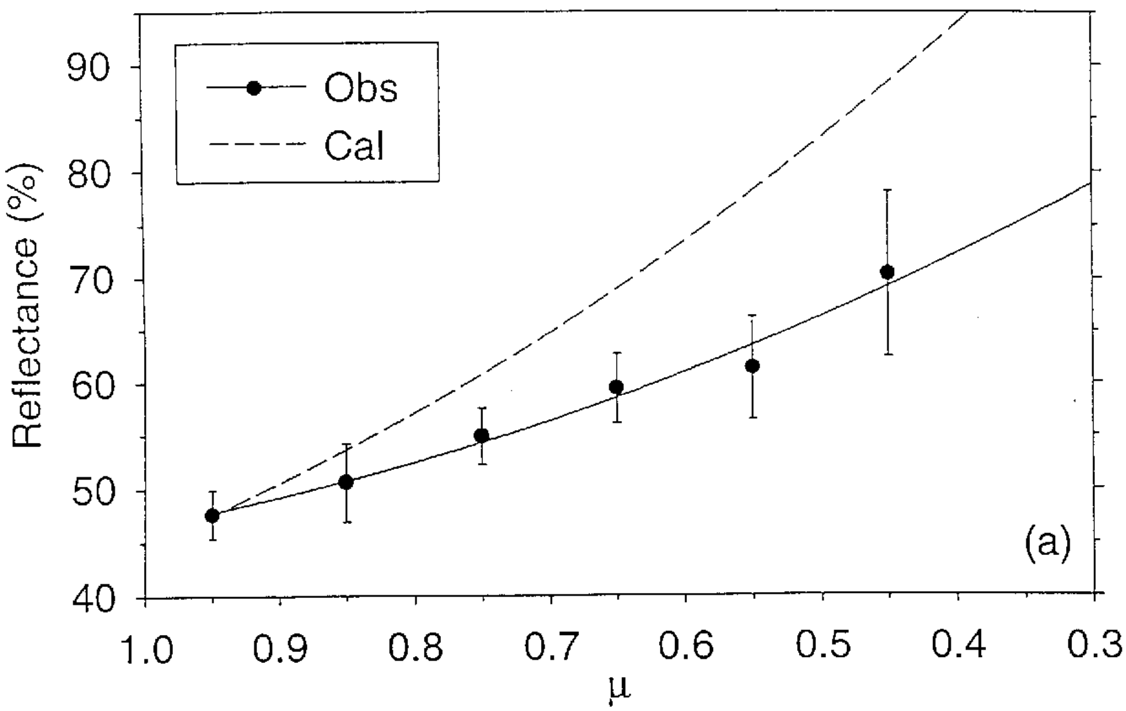
Radiative Smoothing: horizontal diffusion of photons smooths radiances spatially. Scale of smoothing for reflected radiances is $\eta = h/\sqrt{(1-g)\tau}$ where h is geometric thickness.

Side illumination/shadowing by cloud top bumps *increases* radiance variability for large solar zenith angles.



Cloud extinction derived from the NOAA/ETL K-band radar data during the ASTEX experiment (top panel). Nadir reflectance computed with SHDOM for 2D and independent pixels in the 2D extinction field for overhead Sun ($\theta_0 = 0°$) and low Sun ($\theta_0 = 60°$) (lower panels).

Cloud top bumps also appear to cause less reflectance in forward scattering direction. This agrees with observed radiance distributions.



AVHRR reflectance as a function of cosine viewing angle for overcast stratocumulus with solar μ_0 from 0.5 to 0.6 and relative azimuth angle 10° to 30°. For comparison a plane-parallel calculated reflectance curve for optical depth that matches the AVHRR reflectance at $\mu = 1$ is shown.