Multi-Stream Radiative Transfer

Topics:
1. Discrete ordinate form of RTE
2. Doubling-Adding method
3. Surface reflection

Reading: Liou ; Thomas 8.2-8.4,8.11,5.2,6.11

Multi-Stream Radiative Transfer

The two major numerical plane-parallel radiative transfer methods both use discrete ordinates and give the same numerical results.

1. Start with the RTE Fourier transformed in azimuth.
2. Replace the scattering integral by a quadrature sum.
3. The RTE becomes a ordinary differential *matrix* equation.

Radiative transfer equation for each Fourier azimuthal mode m:

$$\begin{aligned} \mu \frac{dI_m(\tau,\mu)}{d\tau} &= I_m(\tau,\mu) - \frac{\omega}{2}\sum_{l=m}^{N} a_{lm}\omega_l \mathcal{P}_l^m(\mu)\int_{-1}^{+1}\mathcal{P}_l^m(\mu')I_m(\tau,\mu')d\mu' \\ &\quad -\frac{\omega}{4\pi}\sum_{l=m}^{N} a_{lm}\omega_l \mathcal{P}_l^m(\mu)\mathcal{P}_l^m(-\mu_0)S_0 e^{-\tau/\mu_0} \end{aligned}$$

Discrete Ordinates and Quadrature Sums

An integral may be approximated by a quadrature sum

$$\int_{-1}^{1} f(\mu)d\mu \approx \sum_{j=1}^{N} w_j f(\mu_j)$$

where μ_j are the *discrete ordinates* and w_j are quadrature weights.

In Gaussian quadrature, μ_j are roots of Legendre polynomial $\mathcal{P}_N(\mu)$.
Weights are $w_j = 2/\{(1-\mu_j^2)[\mathcal{P}'_N(\mu_j)]^2\}$. Weights sum to unity.

Gaussian quadrature is exact for polynomials up to degree $2N-1$.

For plane-parallel RT, double-Gauss quadrature is more accurate.
Double-Gauss: separate quadrature sum for each hemisphere, e.g.

$$\int_0^1 I^{\uparrow}(\mu)d\mu + \int_{-1}^0 I^{\downarrow}(\mu)d\mu \approx \sum_{j=1}^{N} w_j I^{\uparrow}(\mu_j) + \sum_{j=1}^{N} w_j I^{\downarrow}(-\mu_j)$$

where μ_j are Gaussian angles scaled from (-1,+1) to (0,+1).

Discrete Ordinates Radiative Transfer Equation

Discrete ordinates: $2N$ streams, N upward and N downward.

Discrete ordinate RTE

$$\begin{aligned}\pm\mu_j \frac{dI_m(\tau,\pm\mu_j)}{d\tau} &= I_m(\tau,\pm\mu_j) - \frac{\omega}{2}\left[\sum_{j'=1}^{N} w_{j'} P_m^{\pm+}(\pm\mu_j,\mu_{j'}) I_m(\mu_{j'})\right. \\ &\quad \left. + \sum_{j'=1}^{N} w_{j'} P_m^{\pm-}(\pm\mu_j,-\mu_{j'}) I_m(-\mu_{j'})\right] + \mathcal{S}(\pm\mu_j)\end{aligned}$$

The m'th Fourier mode of the phase function is

$$P_m^{\pm+}(\pm\mu_j,\mu_{j'}) = \sum_{l=m}^{N} (2-\delta_{0,m}) \frac{(l-m)!}{(l+m)!} \omega_l \mathcal{P}_l^m(\pm\mu_j) \mathcal{P}_l^m(\mu_{j'})$$

The pluses refer to upwelling, and the minuses to downwelling.

Matrix Form of Radiative Transfer Equation

Radiances at discrete angles are up and down vectors ($\mathbf{I}^+$, $\mathbf{I}^-$).

The matrix RTE is

$$M\frac{d}{d\tau}\begin{pmatrix}\mathbf{I}^+ \\ \mathbf{I}^-\end{pmatrix} = \begin{pmatrix}\mathbf{I}^+ \\ \mathbf{I}^-\end{pmatrix} - \begin{pmatrix}P^{++} & P^{+-} \\ P^{-+} & P^{--}\end{pmatrix}\begin{pmatrix}\mathbf{I}^+ \\ \mathbf{I}^-\end{pmatrix} - \begin{pmatrix}\mathcal{S}^+ \\ \mathcal{S}^-\end{pmatrix}$$

I is radiance vector (one hemisphere of μ_j, one Fourier mode m)
$\mathcal{S}$ is source vector (e.g. diffuse “pseudo-source”)
M is diagonal matrix with $\pm\mu_j$
P is discrete ordinate phase function matrix (with $\omega/2$ and weights w_j).

Reciprocity principle: $P^{++} = P^{--} \quad P^{+-} = P^{-+}$
P^{++} is phase function for upwelling incident and upwelling scattered directions.
P^{+-} is phase function for downwelling incident and upwelling scattered.

Interaction Principle

The RTE is a linear equation in radiance: radiance exitting a layer is linear in radiance incident upon the layer. Represent radiative transfer in a layer with matrix equations.

Interaction principle:

$$I_0^+ = T^+ I_1^+ + R^+ I_0^- + S^+ \qquad I_1^- = T^- I_0^- + R^- I_1^+ + S^-$$

I_0^- and I_1^+ are incident radiance vectors,
T is the transmission matrix,
R is the reflection matrix,
S is the source vector (solar pseudo-source or thermal emission)

Adding Formula

The properties (R and T matrices, S vector) for a combination of two layers can be found from the interaction principle.

The adding formula can be derived from multiple reflections:

$$\begin{aligned}
R_T &= R_1 + T_1 R_2 T_1 + T_1 R_2 R_1 R_2 T_1 + T_1 R_2 R_1 R_2 R_1 R_2 T_1 + \ldots \\
R_T &= R_1 + T_1 R_2 \left[1 + R_1 R_2 + R_1 R_2 R_1 R_2 + \ldots\right] T_1 \\
R_T &= R_1 + T_1 R_2 \left[1 - R_1 R_2\right]^{-1} T_1
\end{aligned}$$

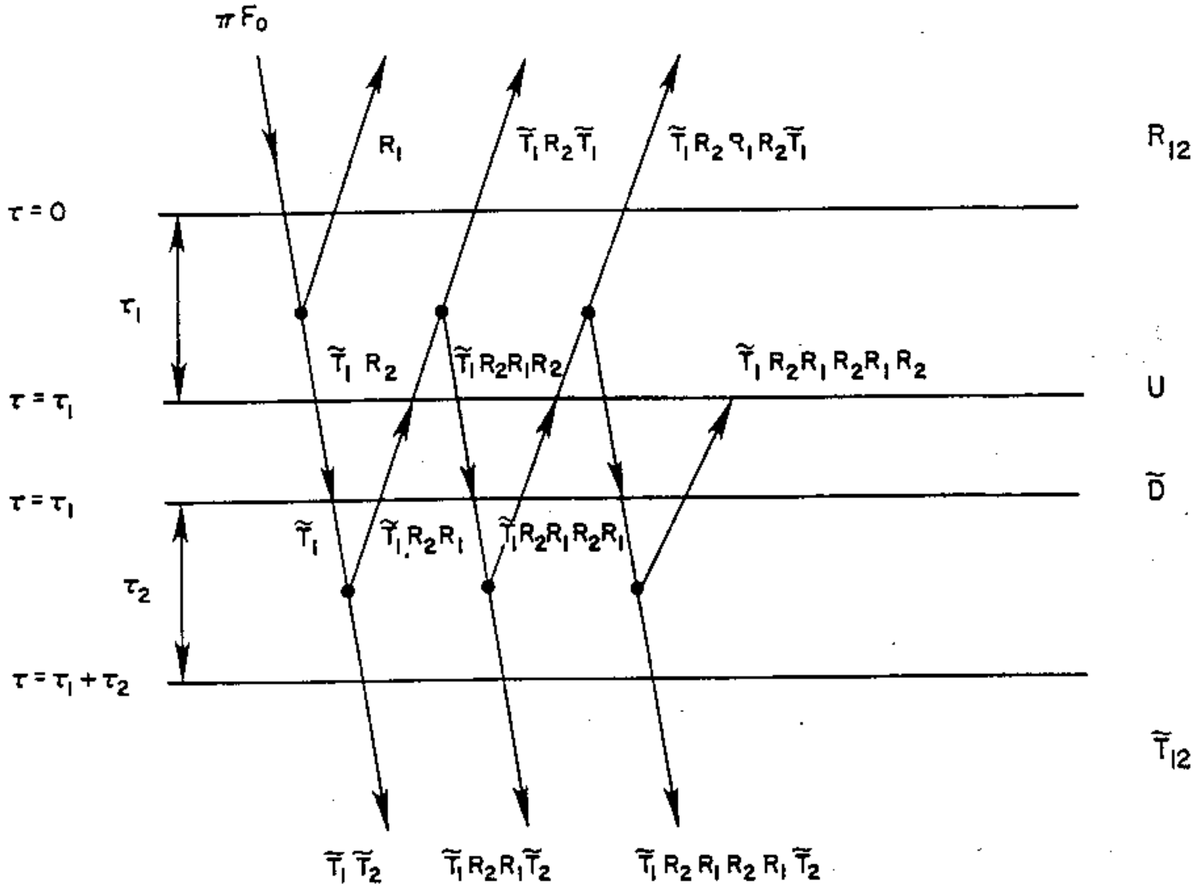
Adding formulas for upwelling radiance (similar for downwelling):

$$R_T^+ = R_1^+ + T_1^+ \Gamma^+ R_2^+ T_1^- \qquad T_T^+ = T_1^+ \Gamma^+ T_2^+$$

$$S_T^+ = S_1^+ + T_1^+ \Gamma^+ (S_2^+ + R_2^+ S_1^-)$$

$$\Gamma^+ = [1 - R_2^+ R_1^-]^{-1}$$

Γ is a multiple reflection factor.



Configuration of the adding method. The two layers of optical depth τ_1 and τ_2 are separated for convenient illustration. [Liou, 1980; Fig. 6.8]

Initialization

Initialization: get R,T,S properties of infinitesimal layer from matrix RTE.

Use finite difference of RTE (optically thin solution) to get properties for layer optical depth $\delta\tau$ (e.g. $\delta\tau = 10^{-5}$).

$$R^{+} = \delta\tau M^{-1} P^{+-} \quad T^{+} = 1 - \delta\tau M^{-1}(1 - P^{++}) \quad S^{+} = \delta\tau M^{-1} \mathcal{S}^{+}$$

Doubling

Doubling: using adding formulas on identical layers.

Doubling formula

$$R^{+}_{2n\delta\tau} = R^{+}_{n\delta\tau} + T^{+}_{n\delta\tau}\Gamma^{+}R^{+}_{n\delta\tau}T^{-}_{n\delta\tau} \qquad T^{+}_{2n\delta\tau} = T^{+}_{n\delta\tau}\Gamma^{+}T^{+}_{n\delta\tau}$$

$$\Gamma^{+} = [1 - R^{+}_{n\delta\tau}R^{-}_{n\delta\tau}]^{-1}$$

For exponential in optical depth source function:

$$S^{+}_{2n\delta\tau} = S^{+}_{n\delta\tau} + T^{+}_{n\delta\tau}\Gamma^{+}(\gamma^{n}S^{+}_{n\delta\tau} + R^{+}_{n\delta\tau}S^{-}_{n\delta\tau})$$

$$S^{-}_{2n\delta\tau} = \gamma^{n}S^{-}_{n\delta\tau} + T^{-}_{n\delta\tau}\Gamma^{-}(S^{-}_{n\delta\tau} + R^{-}_{n\delta\tau}S^{+}_{n\delta\tau}\gamma^{n})$$

where $\gamma = \exp(-\delta\tau/\mu_0)$.

After N doubling steps optical thickness is $2^N\delta\tau$.
What is optical depth after $N = 20$ doubling steps?

Doubling-Adding Method

1. Addition theorem used to calculate Fourier transformed phase function $P_m^{\pm\pm}(\pm\mu_j, \pm\mu_{j'})$ at quadrature angles.
2. Initialization: local reflection R and transmission T matrices for initial layer $\delta\tau$ made from phase function, etc.
3. Doubling: use doubling formula n times to get R, T, S for homogeneous layer with $\Delta\tau = 2^n\delta\tau$.
4. Adding: use adding formula to combine distinct homogeneous layers and surface together (surface has $T = 1, R = R_s, S = 0$).
5. Use interaction principle to apply boundary conditions and obtain outgoing discrete ordinate radiances (or internal radiances).

Surface Reflection

In general, surface reflection can depend arbitrarily on the relation of incident and reflected directions.

Bidirectional Reflectance Distribution Function (BRDF)

$$\rho(\mu, \phi, \mu', \phi') = \frac{I^{\uparrow}(\mu, \phi)}{\mu' I^{\downarrow}(\mu', \phi') d\mu' d\phi'}$$

$\mu' I(\mu', \phi') d\mu' d\phi'$ is flux in solid angle $d\Omega$.

Upwelling radiance is an integral over BRDF and downwelling radiance

$$I^{\uparrow}(\mu, \phi) = \int_0^{2\pi} \int_0^1 \rho(\mu, \phi, -\mu', \phi')\ I^{\downarrow}(-\mu', \phi')\ \mu' d\mu' d\phi'$$

Often surfaces are azimuthally symmetric so BRDF is $\rho(\mu, \mu', \phi - \phi')$.

Albedos

The *albedo* of a surface is the ratio of upwelling to downwelling flux:

$$R_s = \frac{F^{\uparrow}}{F^{\downarrow}} = \frac{\int_0^{2\pi} \int_0^1 \ \int_0^{2\pi} \int_0^1 \rho(\mu, \phi, -\mu', \phi') I^{\downarrow}(-\mu', \phi') \mu' d\mu' d\phi' \ \mu d\mu d\phi}{\int_0^{2\pi} \int_0^1 I^{\downarrow}(-\mu', \phi') \mu' d\mu' d\phi'}$$

Note: in general the albedo *depends on the incident radiance field*.

For collimated incidence the albedo is

$$R_s(\mu_0, \phi_0) = \int_0^{2\pi} \int_0^1 \ \rho(\mu, \phi, -\mu_0, \phi_0)\ \mu d\mu d\phi$$

The *spherical albedo* or *diffuse albedo* is integrated over all collimated incident directions

$$\bar{R}_s = 2 \int_0^1 \ R_s(\mu_0)\ \mu_0 d\mu_0$$

Types of Reflection

A **Lambertian** surface reflects equally in all directions

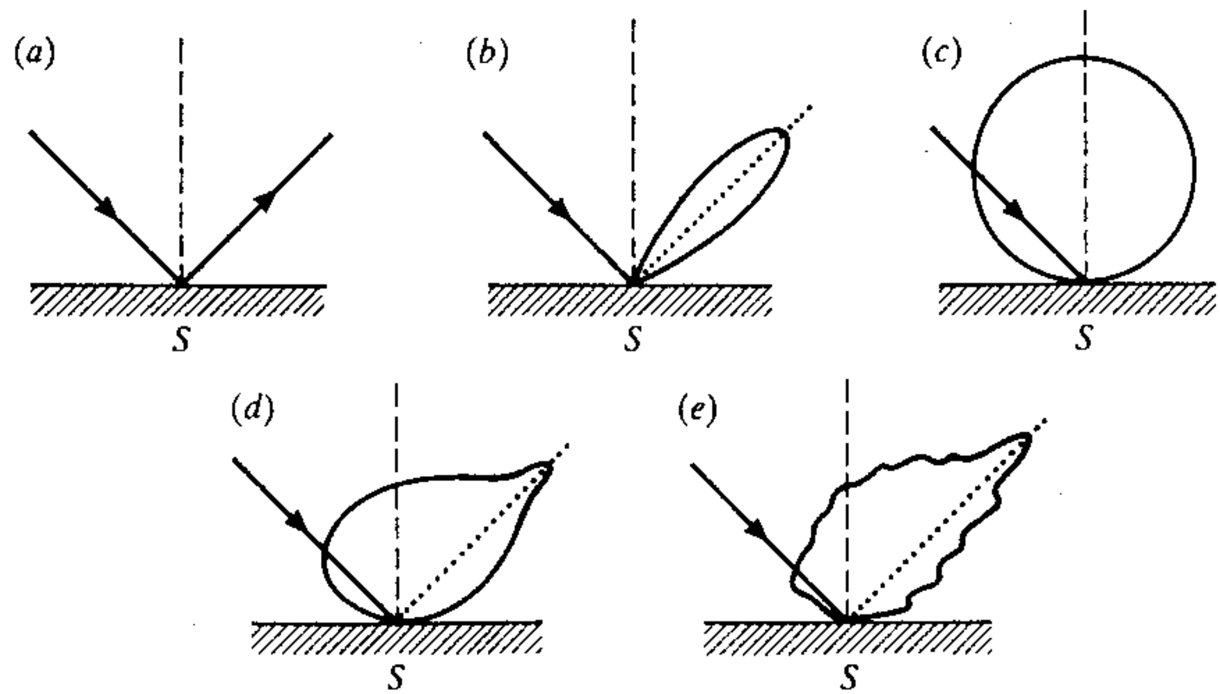
$$\rho(\mu, \phi, \mu', \phi') = \rho_L \qquad I^{\uparrow}(\mu, \phi) = \rho_L F^{\downarrow} = \frac{R_s F^{\downarrow}}{\pi}$$

where R_s is the albedo.

A **specular** surface reflects like a mirror

$$\rho(\mu, \phi, \mu', \phi') = \rho(\mu)\, \delta(\mu + \mu')\delta(\phi - \phi')$$

Real surfaces often have both Lambertian and specular characteristics.



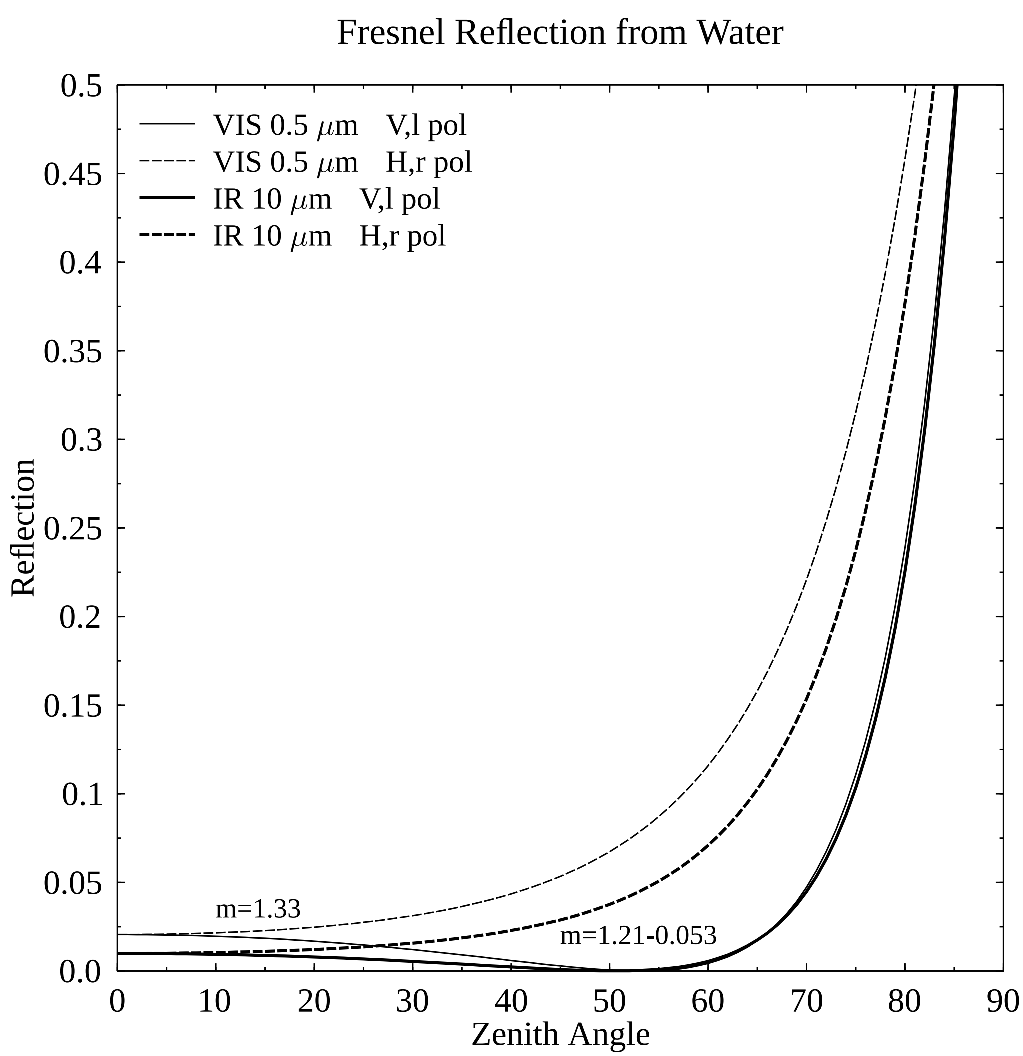
Schematic illustration of different types of surface scattering. The lobes are *polar diagrams* of the scattered radiation. (a) specular; (b) quasi-specular; (c) Labertian; (d) quasi-Lambertian; (e) complex. [Rees, Physical Principles of Remote Sensing, 1990; Fig. 3.2.]

Fresnel Reflection

Fresnel reflection is specular reflection by a smooth dielectric surface.

Fresnel *amplitude* reflection coefficients for each polarization ($r_{\|,V}$ and $r_{\perp,H}$) depend on angle θ and index of refraction m (see week 9 notes).
Intensity reflectance is $R = (|r_{\|}|^2 + |r_{\perp}|^2)/2$.
Reflection at nadir ($\theta = 0$): $R = \left|\frac{m-1}{m+1}\right|^2$.
Completely polarized reflection when $r_{\|,V} = 0$: $\quad \cos\theta_B = \frac{1}{\sqrt{m^2+1}}$
θ_B is called Brewster's or the polarizing angle.

As θ_i increases horizontally polarized reflection increases monotonically; vertically polarized reflection decreases until Brewster angle, then increases. Reflection increases sharply at oblique incident angles.



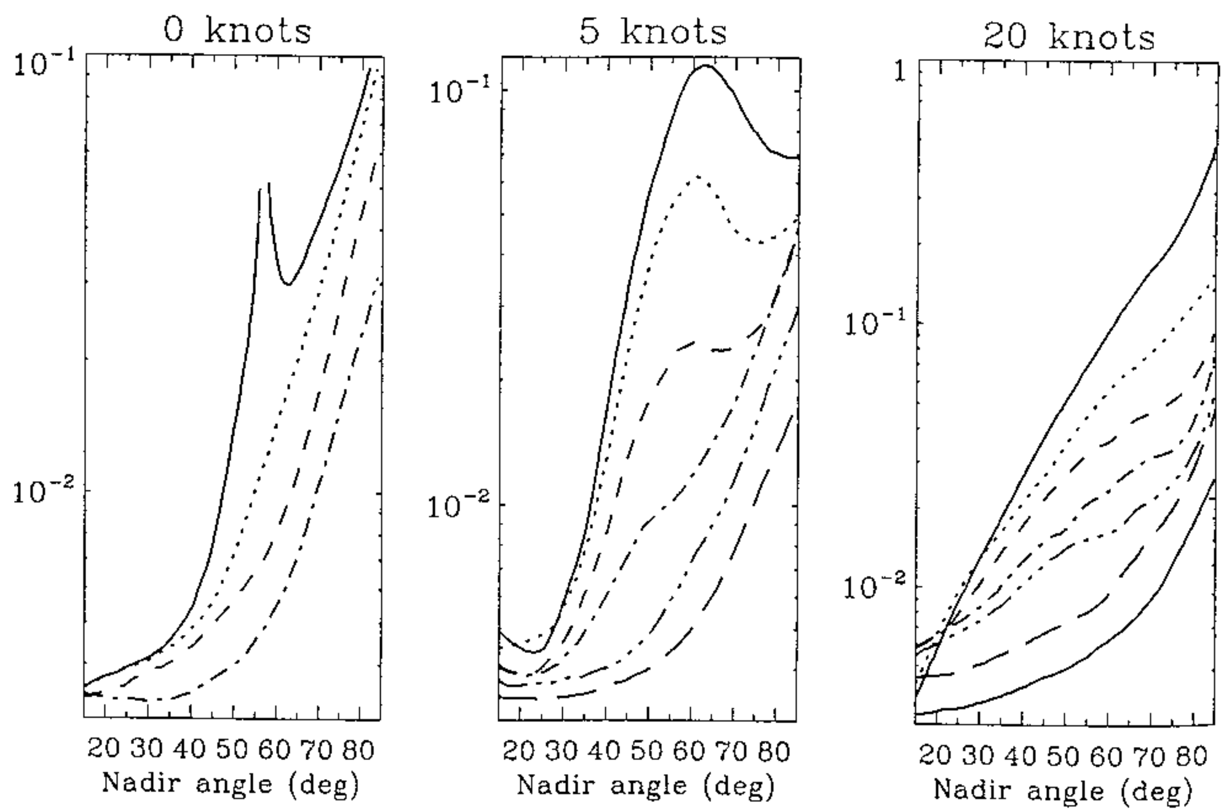
Ocean Reflection

Ocean reflection often modeled by Cox and Munk (1954) model:

wave slope distribution is gaussian with variance proportional to wind speed.
Integrate Fresnel reflection over orientations of facets.

Waves smear out specular peak into broader *sun glint* pattern.

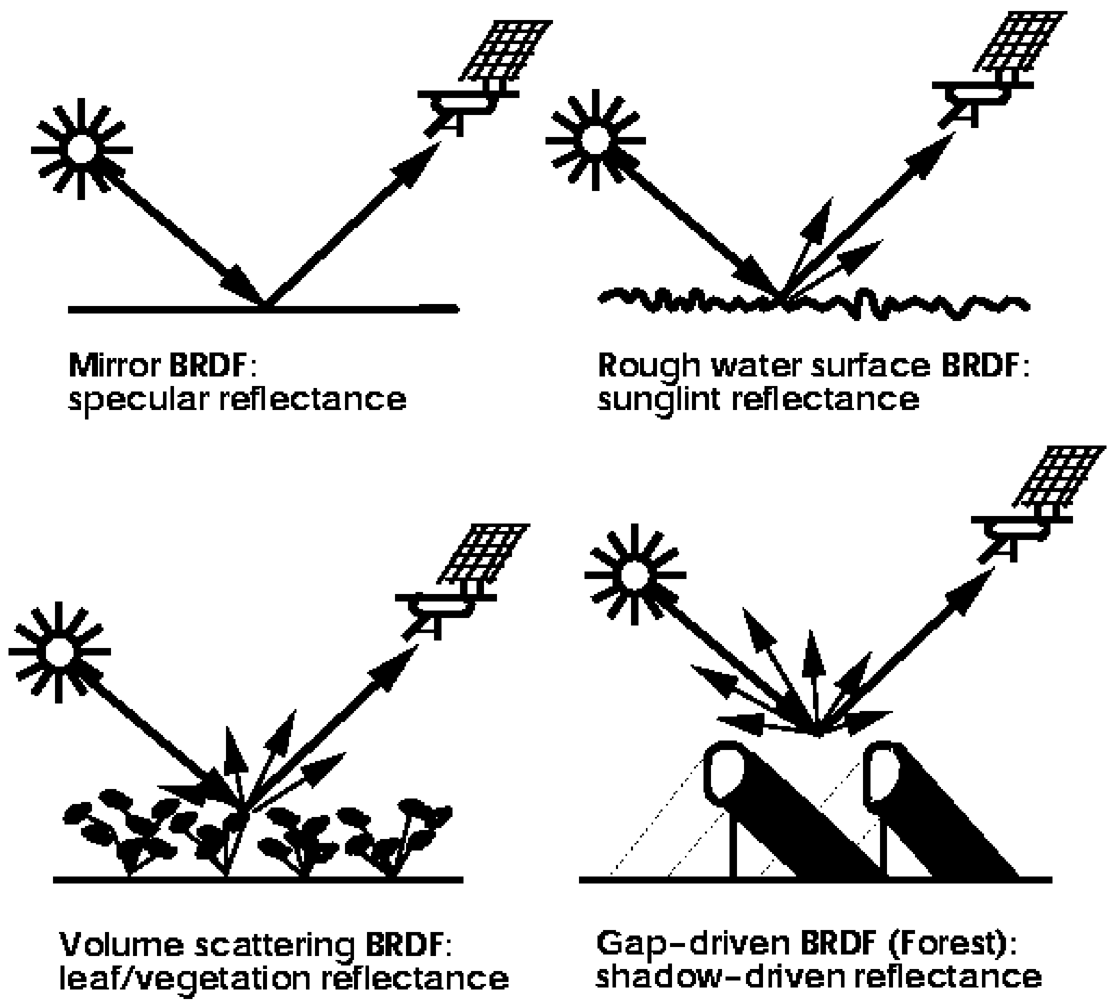
Actual visible reflection from the ocean is usually higher due to whitecaps and scattering from particles in the ocean.



Upward intensity above the ocean surface for $\theta_0 = 57°$ at $\lambda = 0.46\ \mu$m, calculated using a Monte Carlo scheme and the Cox-Munk wave slope distribution. The three plots apply to three different assumed wind speeds. Each curve is for a different azimuthal plane, where $\phi = 0$ being the forward direction in the plane of incidence. [Thomas & Stamnes, Fig. 5.6]

Reflection from soils and vegetation

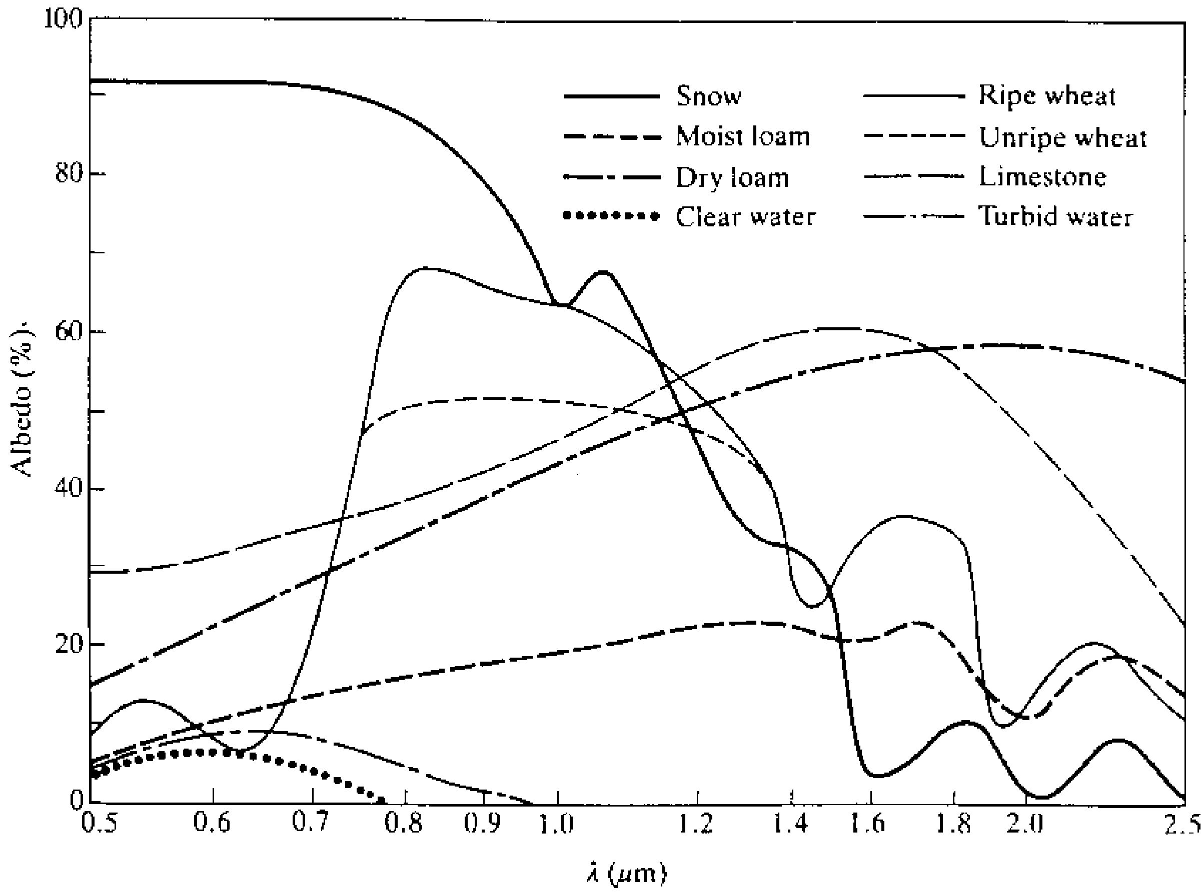
A notable feature of the BRDF for natural surfaces is the *hot spot*:
a peak in reflectance for direct backscatter ($\Theta = 180°$).
Hot spot caused by 1) the lack of observed shadows and
2) specular reflection from oriented leaves.



Basic reasons for land surface reflection anisotropy: specular scattering such as sunglint, also observed where forward-scattering leaves or soil elements are present; radiative transfer-type volumetric scattering by finite scatterers (leaves of plant canopies) that are uniformly distributed, potentially nonuniformly inclined and themselves have anisotropic reflectance; and geometric-optical surface scattering, which is given by shadow-casting and mutual obscuration of three-dimensional surface elements, for example of trees in a sparse forest or brushland, or of clods on a plowed field or of roch-strewn deserts. In natural systems, all types of scattering are likely to occur simultaneously.

Spectral dependence:

1) vegetation has a sharp increase in reflectance around $0.7\ \mu$m (dark in visible, bright in near IR, due to chlorophyl).
2) Soils have a more gradual increase in albedo from VIS to NIR.
3) Snow has high albedo in VIS, decreases in NIR (less albedo in absorption bands and for larger grain size).



Typical spectral albedos (schematic) of various materials in the visible and near infrared bands. [Rees, 1990; Fig. 3.7]

Empirical Reflection Models

Empirical models of surfaces often used to fit observed reflection data.

Minnaert's formula:

$$\rho_M(\mu, \mu_0) = \rho_0 \mu^{k-1} \mu_0^{k-1}$$

A more flexible model from Rahman, Pinty, Verstraete is used for vegetation

$$\rho_{RPV}(\theta_1, \phi_1; \theta_2, \phi_2) = \rho_0 \frac{\cos^{k-1}\theta_1 \cos^{k-1}\theta_2}{(\cos\theta_1 + \cos\theta_2)^{1-k}} F(g)[1 + R(G)]$$

$$F(\Theta) = (1 - g^2)/[1 + g^2 - 2g\cos(\Theta)]^{3/2}$$

$$R(G) = \frac{1 - \rho_0}{1 + G} \quad G = \sqrt{\tan^2\theta_1 + \tan^2\theta_2 - 2\tan\theta_1 \tan\theta_2 \cos(\phi_1 - \phi_2)}$$

which depends on ρ_0, k, g. $F(\Theta)$ controls forward scattering, and $1 + R(G)$ models the hot spot.