Week 10: October 29 - November 2

Two-Stream Radiative Transfer

Topics:
1. Fourier azimuth expansion of RTE
2. Eddington method
3. Delta scaling of RTE
4. Multiple scattering solar flux results
5. Eddington second approximation

Reading: Liou ; Thomas 7.5,6.8

Two-Stream Radiative Transfer

- Two-stream methods (such as Eddington's) provide analytical solutions to the single layer plane-parallel radiative transfer equation.
- There are many related two-stream methods that approximate the angular radiance field with two numbers:
  e.g. constant hemisphere [I^+, I^-], two point quadrature [$I(+\mu_1)$, $I(-\mu_1)$], Eddington - 0'th and 1'st moment [$I(\mu) = I_0 + I_1\mu$].
- These methods are generally only accurate for fluxes.
  However, through a two step process, Eddington's second approximation gives accurate radiances.
- Two-stream methods are used where computational speed is important, such as climate models.

Fourier azimuth series of RTE

The plane-parallel solar radiative transfer equation is

$$\mu \frac{dI(\mu,\phi)}{d\tau} = I(\mu,\phi) - \frac{\omega}{4\pi}\int_0^{2\pi}\int_{-1}^{1} P(\Theta)\; I(\mu',\phi')\; d\mu' d\phi'$$

$$-\frac{\omega}{4\pi} P(\mu,\phi;-\mu_0,\phi_0+\pi)\; S_0 e^{-\tau/\mu_0}$$

where the last term is the pseudosource of diffuse radiation.

Plane-parallel radiative transfer is often solved with a Fourier series in ϕ:

$$I(\tau, \mu, \phi) = \sum_{m=0}^{N} I_m(\tau, \mu) \cos m(\phi_0 - \phi)$$

The $m = 0$ term is the azimuthal average, $I_0(\tau, \mu)$.

Use addition theorem of spherical harmonics for phase function

$$P(\mu, \phi; \mu', \phi') = \sum_{m=0}^{N} \sum_{l=m}^{N} \omega_l \, a_{lm} \, P_l^m(\mu) P_l^m(\mu') \cos m(\phi' - \phi)$$

where $a_{lm} = (2 - \delta_{0,m}) \frac{(l-m)!}{(l+m)!}$ and the Legendre series coefficients ω_l are defined by $P(\cos\Theta) = \sum_{l=0}^{N} \omega_l \mathcal{P}_l(\cos\Theta)$.
This is the major reason for using Legendre series for phase functions.
Substitute the Fourier series for $I(\mu, \phi)$ and the addition theorem phase function in RTE. Scattering integral has $\int \cos m(\phi - \phi') \cos m'(\phi - \phi') d\phi = \delta_{mm'}$ which gets rid of m sum.

Radiative transfer equation becomes (leaving off diffuse source)

$$\mu \frac{dI_m(\tau, \mu)}{d\tau} = I_m(\tau, \mu) - \frac{\omega}{2} \sum_{l=m}^{N} a_{lm} \omega_l P_l^m(\mu) \int_{-1}^{+1} P_l^m(\mu') I_m(\tau, \mu') d\mu'$$

Fourier azimuthal modes separate: N+1 separate equations ($m = 0, \ldots, N$).

Azimuthally Averaged RTE

Fluxes need only $m = 0$ mode:

$$F^{\uparrow}(\tau) = \int_0^{2\pi} \int_0^1 I(\tau, \mu) \, \mu \, d\mu \, d\phi \; = 2\pi \int_0^1 I_{m=0}(\tau, \mu) \mu \, d\mu$$

Azimuthally averaged phase function

$$P(\mu, \mu') = \frac{1}{2\pi} \int_0^{2\pi} P(\mu, \phi; \mu', \phi') d\phi' \; = \sum_{l=0}^{N} \omega_l \, P_l(\mu) P_l(\mu')$$

The azimuthally averaged plane-parallel radiative transfer equation for solar radiation is

$$\mu \frac{dI(\tau, \mu)}{d\tau} = I(\tau, \mu) \; - \; \frac{\omega}{2} \int_{-1}^{1} P(\mu, \mu') I(\tau, \mu') d\mu' \; - \; \frac{\omega}{4\pi} S_0 P(\mu, -\mu_0) e^{-\tau/\mu_0}$$

This is the starting point for two-stream models.

Eddington Method

Approximate radiance field and phase function to first order in μ:

$$I(\tau,\mu) = I_0 + I_1\mu \qquad P(\mu,\mu') = 1 + 3g\mu\mu'$$

The flux at a level is then

$$F^{\uparrow} = \pi\left(I_0 + \frac{2}{3}I_1\right) \qquad F^{\downarrow} = \pi\left(I_0 - \frac{2}{3}I_1\right)$$

Put the radiance and phase function into the radiative transfer equation:

$$\mu\frac{d(I_0 + I_1\mu)}{d\tau} = I_0+I_1\mu - \frac{\omega}{2}\int_{-1}^{1}(1+3g\mu\mu')(I_0+I_1\mu')d\mu' - \frac{S_0\omega}{4\pi}e^{-\tau/\mu_0}(1-3g\mu_0\mu)$$

Doing the scattering integral over μ' results in

$$\mu\frac{d(I_0 + I_1\mu)}{d\tau} = I_0 + I_1\mu \; - \omega(I_0 + I_1 g\mu) \; - \frac{S_0\omega}{4\pi}e^{-\tau/\mu_0}(1 - 3g\mu_0\mu)$$

Rearranging terms gives

$$\mu\frac{dI_0}{d\tau} + \mu^2\frac{dI_1}{d\tau} = I_0(1-\omega) + I_1(1-\omega g)\mu \; - \frac{S_0\omega}{4\pi}e^{-\tau/\mu_0}(1 - 3g\mu_0\mu)$$

First integrate over μ from -1 to 1 and then multiply by μ and integrate from -1 to 1 to get two coupled equations:

$$\frac{dI_1}{d\tau} = 3(1-\omega)I_0 - \frac{3\omega}{4\pi}S_0 e^{-\tau/\mu_0}$$

$$\frac{dI_0}{d\tau} = (1-\omega g)I_1 + \frac{3\omega}{4\pi}g\mu_0 S_0 e^{-\tau/\mu_0}$$

Differentiate the I_0 equation by τ and substitute in $\frac{dI_1}{d\tau}$

$$\frac{d^2 I_0}{d\tau^2} = k^2 I_0 - \frac{3\omega}{4\pi}(1 + g - \omega g)S_0 e^{-\tau/\mu_0}$$

where $k^2 = 3(1-\omega)(1-\omega g)$ is the eigenvalue.
The solutions for I_0 and I_1 are exponential in τ:

$$I_0 = Ce^{k\tau} + De^{-k\tau} + \psi e^{-\tau/\mu_0}$$

where C and D are determined from the boundary conditions.

Eddington Solution Results

The standard boundary conditions for a single layer are no incident radiation from above and below: $F^{\downarrow}(0) = 0$ and $F^{\uparrow}(\tau^*) = 0$.
The solution is quite complicated (e.g. see Meador and Weaver, 1980), so we look at two special cases for a uniform layer of optical depth τ^*.

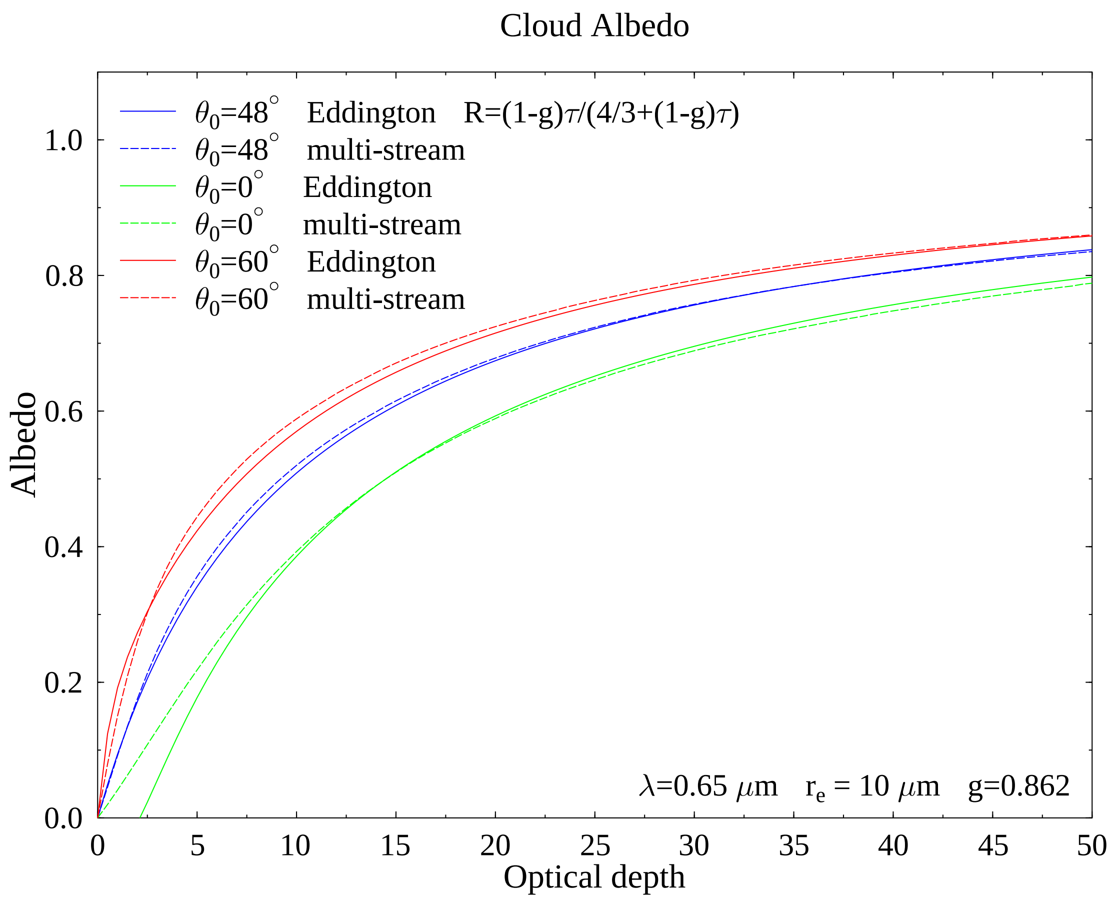
The optically thin solution for reflected and transmitted flux fraction is

$$R = \omega(1/2 - 3g\mu_0/4)\tau^*/\mu_0 \qquad T = 1 - R - (\tau^*/\mu_0)(1-\omega)$$

Problem: negative reflected flux for $g\mu_0 > 2/3$.

For conservative scattering ($\omega = 1$) the fractional reflected flux (albedo) is

$$R = \frac{(1-g)\tau^* + (2/3 - \mu_0)(1 - e^{-\tau^*/\mu_0})}{4/3 + (1-g)\tau^*}$$



Comparison of Eddington and multistream albedo for conservative scattering ($\omega = 1$).

Delta Scaling Transformation

Radiation scattered in the forward diffraction peak is virtually not scattered at all!

- Delta scaling replaces a highly peaked phase function with:
  1) a delta function in the forward direction,
  2) a smoother *scaled* phase function (P').
- Gives a scaled radiative transfer equation with less optical depth.

Delta scaling of phase function with forward scattering fraction f:

$$P(\cos\Theta) \approx 2f\delta(1-\cos\Theta) + (1-f)P'(\cos\Theta)$$

The scaled phase function should have the same asymmetry parameter

$$g = \frac{1}{2}\int_{-1}^{1} P(\cos\Theta)\cos\Theta \, d\cos\Theta = f + (1-f)g'$$

Delta Scaling of the Radiative Transfer Equation

Can delta scale any form of the radiative transfer equation.

Azimuthally averaged RTE:

$$\mu\frac{dI}{d\tau} = I(\tau,\mu) - \frac{\omega}{2}\int_{-1}^{1} P(\mu,\mu')I(\tau,\mu')d\mu' + S(\tau,\mu)$$

Put in our approximation for the true phase function

$$\mu\frac{dI}{d\tau} = I - f\omega I - \frac{1}{2}(1-f)\omega\int_{-1}^{1} P'(\mu,\mu')I(\mu')d\mu' + S'$$

$$\mu\frac{dI}{(1-\omega f)d\tau} = I - \frac{1}{2}\frac{(1-f)\omega}{1-\omega f}\int_{-1}^{1} P'(\mu,\mu')I(\mu')d\mu' + S'$$

This is the same radiative transfer equation if we scale variables!

$$\tau' = (1-\omega f)\tau \quad \omega' = \frac{(1-f)\omega}{1-\omega f} \quad g' = \frac{g-f}{1-f}$$

Procedure: delta scale extinction ($\beta' = (1-\omega f)\beta$), single scattering albedo, and phase function, then use in regular radiative transfer equation.

How does scaling change the optical properties?

How do we get the delta scaling fraction f?

Delta-isotropic: make scaled phase function isotropic $g' = 0$: $f = g$

Delta-Eddington: make two term scaled phase function:
Choose $f = \omega_2/5$ so second moment is preserved.
If ω_2 is unavailable, get it from Henyey-Greenstein $f = g^2$.

$$\tau' = (1 - \omega g^2)\tau \quad \omega' = \frac{(1-g^2)\omega}{1-\omega g^2} \quad g' = \frac{g}{1+g}$$

Delta-M Approximation: gives accurate fluxes in numerical radiative transfer models with M discrete "streams" per hemisphere:

$$\chi'_l = \frac{\chi_l - f}{1-f} \quad l < 2M \quad f = \chi_{2M} \qquad \chi_l = \frac{\omega_l}{2l+1}$$

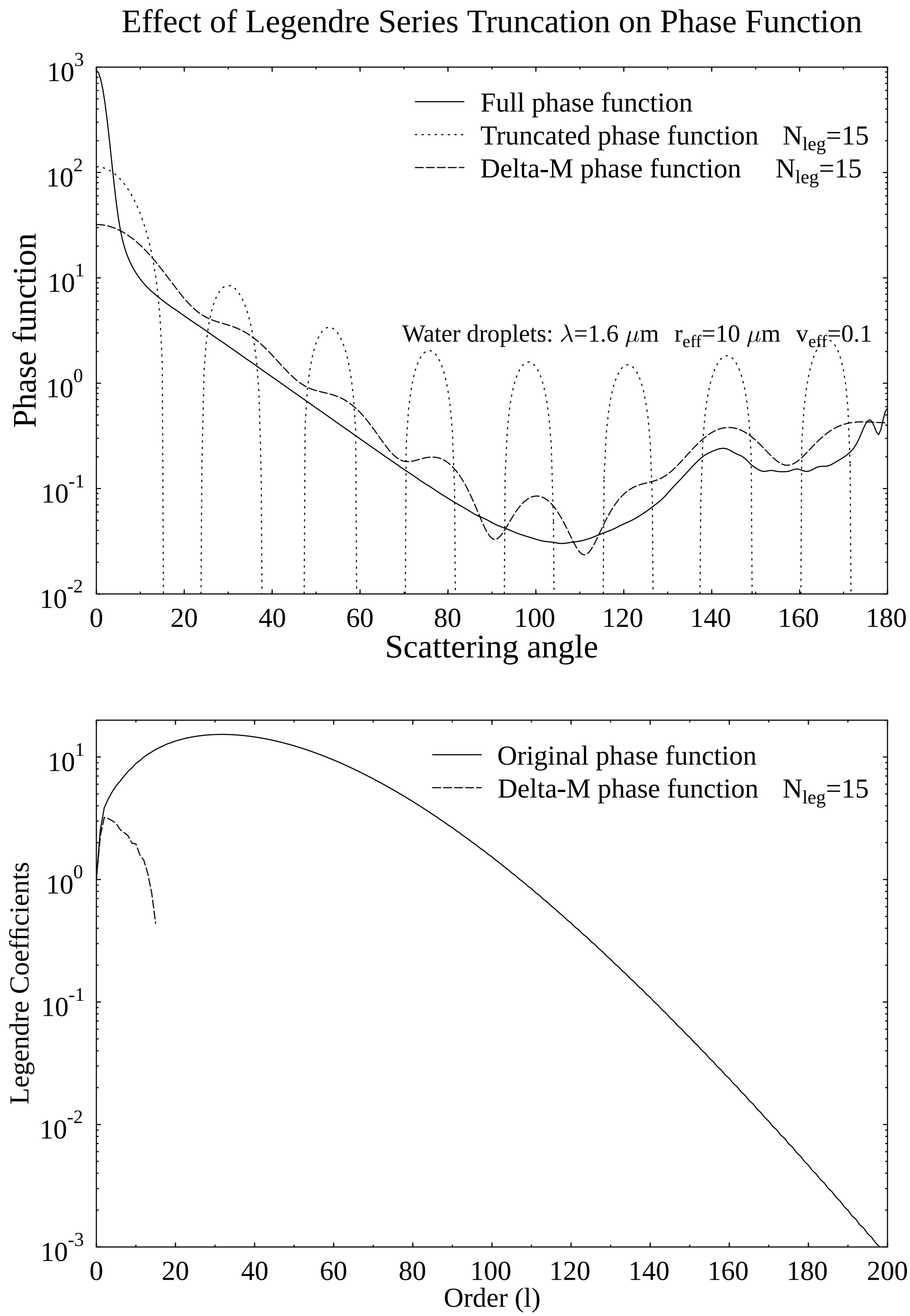
Delta-Eddington is case with $M = 1$.
Delta-M phase function has much less oscillation than truncated phase function ($\omega_l = 0$ for $l > 2M$).

Delta Scaling Summary

Scaled radiative transfer system has:
Lower optical depth + less forward scattering = same reflection, etc.

Similarity principle: τ, ω, g is equivalent to τ', ω', g'

Lower scaled optical depth $\rightarrow$ higher direct beam transmission.
Must add diffuse and direct transmitted flux to get "correct" total.



Comparison of a Mie phase function for cloud droplets with the phase function of the Legendre series truncated at order $l = 15$ without and with delta-M scaling applied (top). The original and delta-M scaled Legendre series coefficients of the phase function (bottom).

Multiple Scattering Flux Reflection Results

Fundamental property of reflectivity from radiative transfer:

Linear for $\tau \ll 1$ (first order solution)

Saturation for $\tau \gg 1$

More forward scattering means less reflection ($g \uparrow \;\Rightarrow\; R \downarrow$)

Equivalent isotropic scattering optical depth: $\tau' = (1 - \omega g)\tau$

Higher solar zenith angle means more reflection unless optically thin: ($\mu_0 \downarrow \;\Rightarrow\; R \uparrow$)

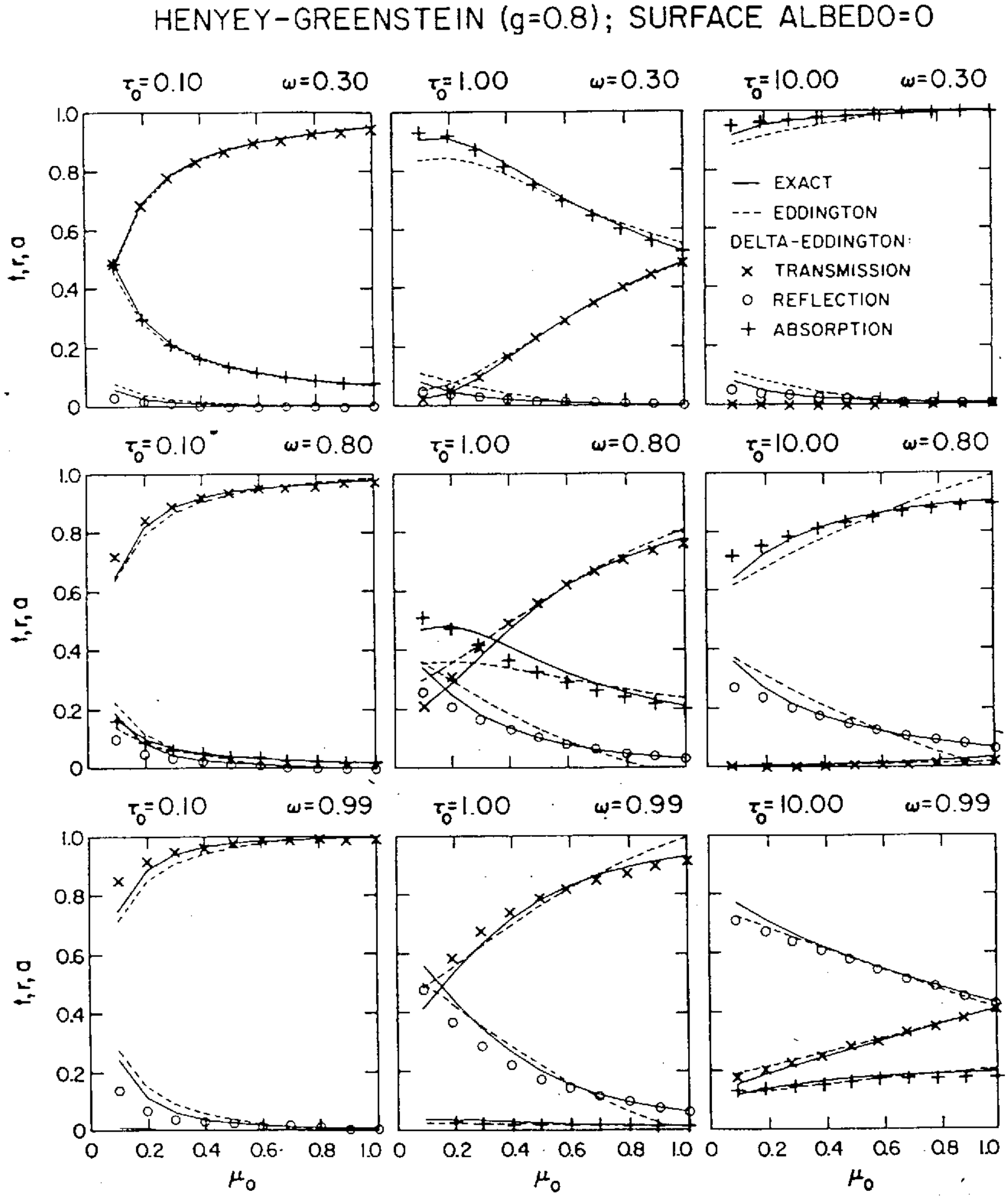
Multiple scattering amplifies absorption ($\mu_0 = 2/3 \;\; g = 0.85$):

$\tau = 1 \quad \omega = 0.99 \qquad R = 0.096 \; A = 0.018$

$\tau = 10 \quad \omega = 0.99 \qquad R = 0.45 \;\; A = 0.18$

$\tau = 100 \;\; \omega = 0.99 \qquad R = 0.55 \;\; A = 0.45$

$\tau = 100 \;\; \omega = 1.00 \qquad R = 0.92 \;\; A = 0.00$



Reflectivity (r), transmissivity (t), and absorptivity (a) as a function of cosine of solar zenith angle (μ_0) for various single-scattering albedoes (ω) and later optical depths (τ_0), comparing exact, Eddington and delta-Eddington methods for asymmetry factor $g = 0.8$ and surface albedo ($A = 0$). [Joseph and Wiscombe, 1976: The Delta-Eddington Approximation for Radiative Flux Transfer, J. Atmos. Sci., 33, 2452.]

Eddington Second Approximation for Radiances

The Eddington approximation can give accurate radiances only through a two-step process:

1) The Eddington solution gives the crude radiance field,
2) The source function, with scattering, is found from the Eddington I_0,I_1, and then integrated for the radiance.

Used for thermal emission with scattering, where source function is

$$J(\tau, \mu) = \frac{\omega}{2} \int_{-1}^{1} P(\mu, \mu') I(\tau, \mu') d\mu' + (1 - \omega) B(T)$$

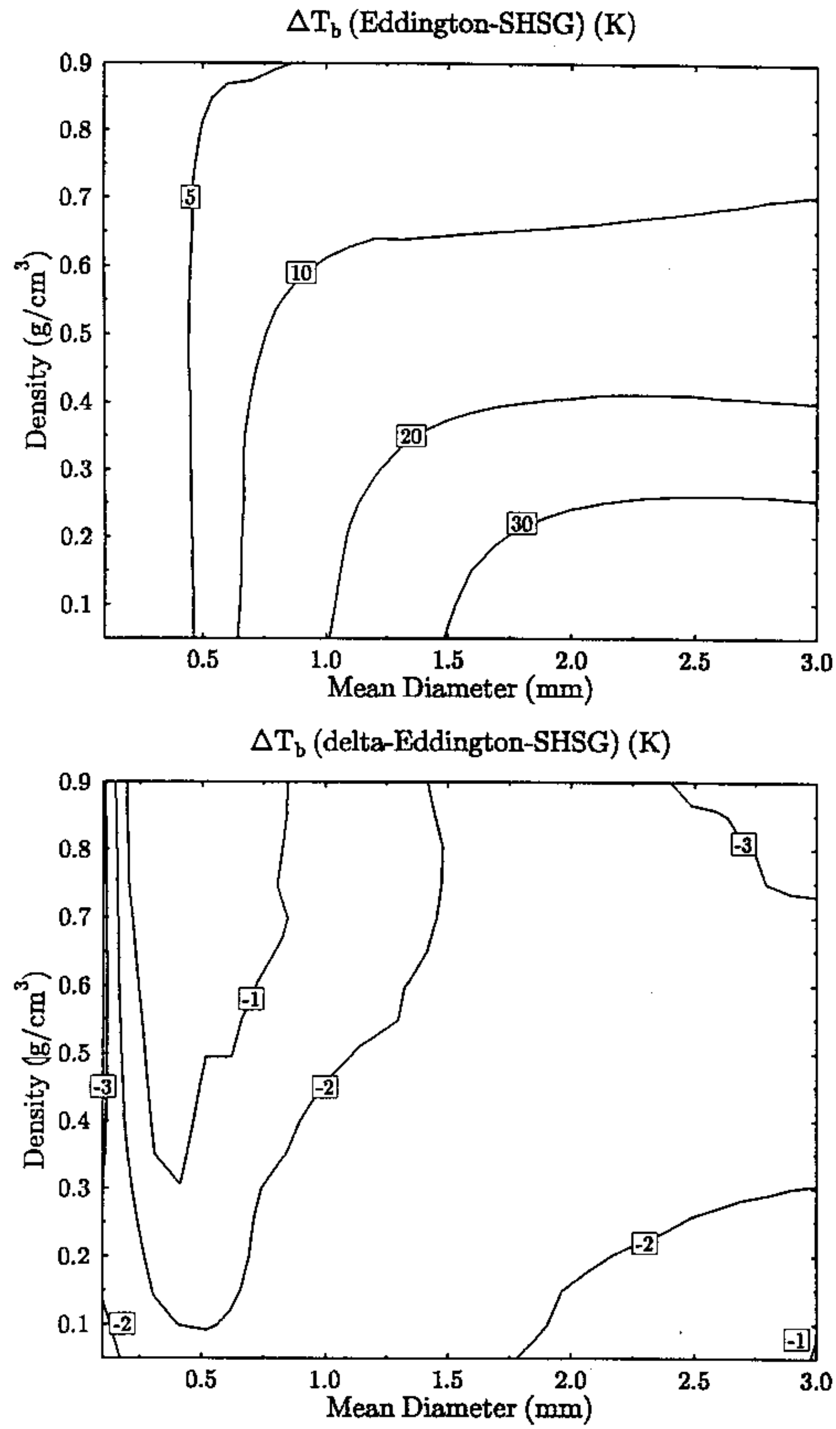
Putting in the Eddington approximation:

$$J_{edd}(\mu) = \omega(I_0 + I_1 g\mu) + (1 - \omega) B(T)$$

Upwelling radiance at top is then

$$I^{\uparrow}(\tau = 0, \mu) \ = \int_0^{\tau_*} J_{edd}(\tau, \mu) \, e^{-\tau/\mu} \, d\tau/\mu$$

Eddington's second approximation works well when scattering integral is like a low order moment.



Difference in upwelling zenith brightness temperature between the Eddington approximation and spherical harmonic ($L = 11$) radiative transfer methods at 85.5 GHz for a modeled ice particle layer. There is a single uniform ice sphere layer of optical depth 2 at 85.5 GHz with temperature from 270 to 245 K above a blackbody surface at 270 K. The top panel is for unscaled Eddington, and the bottom is for delta-scaled Eddington. [Evans, 1993, PhD thesis]