ASTR/ATOC 5560 Problem Solving Solutions Week 7

1. *For large size parameters, say $x > 50$, the extinction efficiency asymptotes to $Q_{ext} = 2$. Relate the volume extinction coefficient, β_{ext}, of a particle size distribution to the liquid water content (LWC) and the effective radius.*

   The volume extinction coefficient for a particle size distribution is

   $$\beta_{ext} = \int_0^\infty C_{ext}\, n(r) dr \; = \int_0^\infty \pi r^2 Q_{ext}\, n(r) dr$$

   So for $Q_{ext} = 2$ the extinction is proportional to the second moment of the size distribution

   $$\beta_{ext} = 2\pi \int_0^\infty r^2\, n(r) dr$$

   The liquid water content is proportional to the third moment, and the effective radius is (by design) the ratio of the third to the second moment. Thus we take the ratio of LWC to effective radius

   $$\frac{W}{r_e} = \frac{\frac{4\pi}{3}\rho_l \int r^3 n(r) dr}{\int r^3 n(r) dr / \int r^2 n(r) dr} \; = \frac{2\rho_l}{3}\left[2\pi \int r^2 n(r) dr\right]$$

   Solving for the extinction gives

   $$\beta_{ext} = \frac{3W}{2\rho_l r_e}$$

   which is true for large size parameters.

2. *Use the above relation to determine the extinction of a lognormal droplet distribution with $N = 10\ \mathrm{cm}^{-3}$, $r_0 = 2\ \mu\mathrm{m}$, and $\sigma = 0.4$.*

   For a lognormal distribution the effective radius and mass content are

   $$r_{\text{eff}} = r_0 \exp(2.5\sigma^2) \qquad W = \frac{4\pi}{3}\rho_l N r_0^3 \exp(4.5\sigma^2)$$

   Using the above relation for extinction

   $$\beta_{ext} = \frac{3W}{2\rho_l r_e} \; = \frac{4\pi\rho_l N r_0^3 \exp(4.5\sigma^2)}{2\rho_l r_0 \exp(2.5\sigma^2)} = 2\pi N r_0^2 \exp(2\sigma^2)$$

   It is simpler to find the second moment of a lognormal distribution from

   $$\int_0^\infty r^k n(r) dr = N r_0^k \exp(k^2\sigma^2/2)$$

   and relate this to the extinction in the large size parameter limit:

   $$\beta_{ext} = 2\pi N r_0^2 \exp(2\sigma^2)$$

   For the size distribution given, the extinction is

   $$\beta_{ext} = 2\pi(10\ \mathrm{cm}^{-3})(2\times10^{-4}\ \mathrm{cm})^2 \exp(0.32) \; = 3.46\times10^{-6}\ \mathrm{cm}^{-1} = 0.35\ \mathrm{km}^{-1}$$