# ASTR/ATOC 5560 Problem Solving Solutions Week 4

1. *Between 691 cm$^{-1}$ and 693 cm$^{-1}$ the HITRAN 2000 database contains the following parameters for lines of the relevant gases with line strengths above $10^{-21}$ cm/molecule:*

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31  691.370100 1.040E-21 8.869E-04.0761.1003  228.7342 .76
21  691.972498 9.167E-20 9.167E-03.0684.0869  362.7882 .78
31  692.034800 1.160E-21 9.060E-04.0767.1009  191.7092 .76
21  692.129032 4.010E-21 9.500E-03.0686.0878 1007.1335 .78
21  692.400132 3.685E-21 9.484E-03.0684.0869 1031.1291 .78
31  692.468700 1.240E-21 9.110E-04.0771.1016  158.1653 .76
31  692.661600 1.070E-21 7.898E-04.0780.1038   78.5386 .76
31  692.696400 1.240E-21 8.823E-04.0775.1023  128.1196 .76
31  692.748400 1.190E-21 8.469E-04.0778.1030  101.5791 .76
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*We would like to compute the optical depth at a wavenumber of 692.000 cm$^{-1}$ for a 1 km thick layer at a pressure of $p = 102$ mb and temperature of $T = 217$ K. The volume mixing ratios of the radiatively active gases are $q_{H2O} = 6.55 \times 10^{-6}$, $q_{CO2} = 3.70 \times 10^{-4}$, and $q_{O3} = 2.22 \times 10^{-6}$. The number of air molecules per cm$^2$ in this layer is*

$$u_{air} = \frac{\Delta z N_A p}{R^* T} = 3.40 \times 10^{23}\ \mathrm{molec/cm^2} \quad .$$

*a) Determine the single absorption line that will dominate the absorption at 692.000 cm$^{-1}$. Write down the molecule and wavenumber.*

The important absorption line is the one at 691.9725 cm$^{-1}$ which is from the main isotope of $CO_2$. The other $CO_2$ lines have lower strengths and are much further in terms of line halfwidths from 692 cm$^{-1}$. The nearby ozone line is not relevant since the ozone abundance is 167 times less than $CO_2$ and the line strength at 296 K is about 80 times less.

*b) Calculate the Lorentz halfwidth for this absorption line. Self-broadening can be ignored since the mixing ratios are so small.*

The halfwidth of the Lorentz line shape scales according to

$$\alpha = \alpha_0 \left(\frac{p}{p_0}\right)\left(\frac{T_0}{T}\right)^n$$

where $\alpha_0$ is the halfwidth in the HITRAN database at pressure $p_0 = 1013.25$ mb and temperature $T_0 = 296$ K. The halfwidth of this line is

$$\alpha = (.0684\ \mathrm{cm}^{-1})\left(\frac{102\ \mathrm{mb}}{1013\ \mathrm{mb}}\right)\left(\frac{296\ \mathrm{K}}{217\ \mathrm{K}}\right)^{0.78} = 0.00877\ \mathrm{cm}^{-1}$$

*c) Calculate the line stength for this absorption line at the temperature of this layer. The partition functions are*

| *Molecule/Isotope* | *Q(217 K)* | *Q(296 K)* |
|---|---|---|
| *21* | *198.4* | *286.2* |
| *31* | *2094.4* | *3481.9* |

The line strength scales as

$$S_i(T) = S_i(T_0)\frac{\exp(-hcE_{L,i}/k_BT)Q_T(T_0)\left[1-\exp(-hc\nu_{0,i}/k_BT)\right]}{\exp(-hcE_{L,i}/k_BT_0)Q_T(T)\left[1-\exp(-hc\nu_{0,i}/k_BT_0)\right]}$$

where $T$ is the temperature, $T_0$ is the reference temperature (296 K), $Q_T$ is the total partition function for the particular molecule/isotope, $E_{L,i}$ is the lower state energy in cm$^{-1}$, and $\nu_{0,i}$ is the transition wavenumber.

For this $CO_2$ absorption line the line strength is

$$S_i(T) = S_i(T_0)\,\frac{\exp\left[-\frac{(1.4388\text{ K cm})(362.7882\text{ cm}^{-1})}{217\text{ K}}\right]}{\exp\left[-\frac{(1.4388\text{ K cm})(362.7882\text{ cm}^{-1})}{296\text{ K}}\right]}\,\frac{286.2}{198.4}\,\frac{\left(1-\exp\left[-\frac{(1.4388\text{ K cm})(691.9725\text{ cm}^{-1})}{217\text{ K}}\right]\right)}{\left(1-\exp\left[-\frac{(1.4388\text{ K cm})(691.9725\text{ cm}^{-1})}{296\text{ K}}\right]\right)}$$

$$S_i(T) = (9.167\times10^{-20}\text{ cm/molec})\left(\frac{0.090}{0.171}\right)(1.443)\left(\frac{0.9898}{0.9654}\right)$$

$$S_i(T) = (9.167\times10^{-20}\text{ cm/molec})(0.526)(1.443)(1.025)\ = 7.136\times10^{-20}\text{ cm/molec}$$

*d) Calculate the absorption coefficient and optical depth of this layer at 692 cm$^{-1}$.*

Since we are considering a single Lorentz absorption line, we use the Lorentz line shape $f$. The the absorption coefficient is given by

$$k_\nu = Sf(\nu-\nu_0) = \frac{S\alpha/\pi}{(\nu-\nu_0)^2+\alpha^2}$$

For the desired wavenumber of $\nu = 692.000\text{ cm}^{-1}$ the monochromatic absorption coefficient is

$$k_\nu = \frac{(7.136\times10^{-20}\text{ cm/molec})(0.00877\text{ cm}^{-1})/\pi}{(692.0000\text{ cm}^{-1}-691.9725\text{ cm}^{-1})^2+(0.00877\text{ cm}^{-1})^2}\ = 2.39\times10^{-19}\text{ cm}^2\text{/molec}$$

The optical depth of the layer at this wavenumber is the absorber amount of $CO_2$ times the absorption coefficient

$$\tau_n u = k_\nu u_{CO2} = k_\nu q_{CO2} u_{air}$$

$$\tau_n u = (2.39\times10^{-19}\text{ cm}^2\text{/molec})(3.70\times10^{-4})(3.40\times10^{23}\text{ molec/cm}^{-2})\ = 30.1$$