ASTR/ATOC 5560 Problem Solving Solutions Week 1

1. *In the Eddington approximation the radiance field is expanded to first order as* $I(\mu) = I_0 + \mu I_1$ *(no azimuthal dependence). What is the upwelling and downwelling hemispheric flux, the net flux, and the actinic flux, in terms of* I_0 *and* I_1*?*

The upwelling flux is the cosine weighting of the upwelling radiance:

$$F^{\uparrow} = \int_0^{2\pi} \int_0^1 I(\mu, \phi)\, \mu \, d\mu \, d\phi \;\; = \;\; \int_0^{2\pi} \int_0^1 (I_0 + \mu I_1) \mu \, d\mu \, d\phi$$

$$F^{\uparrow} = 2\pi \left[I_0 \frac{1}{2} \mu^2 + I_1 \frac{1}{3} \mu^3 \right]_0^1 \;\; = \pi I_0 + \frac{2\pi}{3} I_1$$

The downwelling flux is

$$F^{\downarrow} = \int_0^{2\pi} \int_{-1}^0 I(\mu, \phi)\, |\mu| \, d\mu \, d\phi \;\; = \;\; - \int_0^{2\pi} \int_0^1 (I_0 + \mu I_1) \mu \, d\mu \, d\phi$$

The cosine weighting factor must be positive so we assure this with the absolute value sign.

$$F^{\downarrow} = -2\pi \left[I_0 \frac{1}{2} \mu^2 + I_1 \frac{1}{3} \mu^3 \right]_{-1}^0 \;\; = \pi I_0 - \frac{2\pi}{3} I_1$$

Note that if the radiation is isotropic, $I_1 = 0$, then the upwelling and downwelling flux are the same, namely πI_0.
The net flux is the difference between upwelling and downwelling hemispheric flux:

$$F_{net} = \int_0^{2\pi} \int_{-1}^1 I(\mu, \phi)\, \mu \, d\mu \, d\phi \; = F^{\uparrow} - F^{\downarrow} \; = \frac{4\pi}{3} I_1$$

The net flux is proportional to I_1. The upwelling and downwelling hemispheric fluxes are positive, but the net flux may be positive or negative depending on whether the net energy flow is upwards or downwards.
The actinic flux is the integral of the radiance over all angles without the cosine weighting:

$$F_0 = \int_0^{2\pi} \int_{-1}^1 I(\mu, \phi) \, d\mu \, d\phi \; = \int_0^{2\pi} \int_{-1}^1 (I_0 + \mu I_1) \, d\mu \, d\phi$$

$$F_0 = 2\pi \left[I_0 \mu + I_1 \frac{1}{2} \mu^2 \right]_{-1}^1 \;\; = 4\pi I_0$$

The actinic flux is proportional to I_0, which is the mean radiance.

2. *The emissivity of quartz at 9.5 μm is about 0.5. Calculate the radiance at 9.5 μm of quartz at 230 K. Then calculate the brightness temperature.*

The emitted radiance is

$$I_\lambda = \epsilon_\lambda B_\lambda(T)$$

The Planck function is

$$B_\lambda(T) = \frac{c_1}{\lambda^5[\exp(c_2/\lambda T) - 1]}$$

$$c_1 = 1.1911 \times 10^8 \text{ W m}^{-2} \text{ sr}^{-1} \ \mu\text{m}^4 \quad c_2 = 1.4388 \times 10^4 \text{ K } \mu\text{m}$$

$$B_\lambda(T) = \frac{1.1911 \times 10^8 \text{ W m}^{-2} \text{ sr}^{-1} \ \mu\text{m}^4}{(9.5 \ \mu\text{m})^5[\exp(1.4388 \times 10^4 \text{ K } \mu\text{m}/(9.5 \ \mu\text{m})(230 \text{ K})) - 1]}$$

$$B_\lambda(T) = 2.13 \text{ W m}^{-2} \text{ sr}^{-1} \ \mu\text{m}^{-1}$$

Therefore the emitted radiance is

$$I_\lambda = 0.5(2.13 \text{ W m}^{-2} \text{ sr}^{-1} \ \mu\text{m}^{-1}) = 1.06 \text{ W m}^{-2} \text{ sr}^{-1} \ \mu\text{m}^{-1}$$

The brightness temperature corresponding to this radiance is

$$T_b = \frac{c_2}{\lambda \ln[1 + c_1/(I_\lambda \lambda^5)]}$$

$$T_b = 208.1 \text{ K}$$

Note that this is much larger than half the physical temperature. For what wavelengths would the brightness temperature for an emissivity of 0.5 actually be half the physical temperature?