

## ATOC/ASTR 5560 Lab 1 Solution

The purpose of this computational lab is to gain familiarity with the Planck function and its integration across the spectrum. Log in to nit and copy the IDL file to your directory:

```
cp /home/rt/planck/planck.pro .
```

1. At the beginning of the IDL file are function definitions for the two Planck functions. Fill in the Planck expressions in the two functions.

See the solution IDL procedure in /home/rt/planck/plancksol.pro.

2. Use section PlanckSpectrum to make spectrum plots of the Planck function per wavelength and per wavenumber for temperatures of 200, 250, and 300 K.

See plot in /home/rt/planck/PlanckSpectrum.ps.

a) Write down the wavelength/wavenumber and Planck function value at the maximum for each temperature. The IDL max function might be helpful, e.g.

```
print, max(Blambda[*],0), lamdagrid[im]
```

See if the results agree with Wien's displacement law.

Maximum wavelength:

Temperature	IDL calculation		Wien's
$T$ (K)	$\lambda_{max}$ ( $\mu\text{m}$ )	$B_\lambda$ ( $\text{W m}^{-2} \text{sr}^{-1} \mu\text{m}^{-1}$ )	$\lambda_{max}$ ( $\mu\text{m}$ )
300	9.65	9.95	9.67
250	11.60	4.00	11.60
200	14.50	1.31	14.50

Maximum wavenumber:

Temperature	IDL calculation	
$T$ (K)	$\nu_{max}$ ( $\mu\text{m}$ )	$B_\nu$ ( $\text{W m}^{-2} \text{sr}^{-1} \text{cm}$ )
300	590	0.1535
250	490	0.0888
200	390	0.0455

b) What is the dependence of the maximum Planck function value on temperature for both  $B_\nu$  and  $B_\lambda$  (e.g. the exponent in  $B_\nu \propto T^n$ )? Use the 200 K and 300 K Planck values to determine this.

Use the 200 K and 300 K temperatures and take the ratio of  $B_\nu$  and  $T$ :

$$\frac{B_{\nu,1}}{B_{\nu,2}} = \left(\frac{T_1}{T_2}\right)^n$$

Then take logs to get  $n = \ln(B_{\nu,1}/B_{\nu,2})/\ln(T_1/T_2)$ . For  $B_\nu$ ,  $n = 3$ . For  $B_\lambda$ ,  $n = 5$ .

c) Do the maximums in terms of wavenumber correspond to those in wavelength (i.e. convert the  $\nu_{max}$  to wavelength)? Why or why not?

No. For example, for  $T = 300$  K,  $\nu_{max} = 590 \text{ cm}^{-1}$  corresponds to a wavelength of  $\lambda = 16.9 \text{ }\mu\text{m}$  instead of the maximum of  $B_\lambda$  of  $9.67 \text{ }\mu\text{m}$ . This is because the two Planck functions are different functions, related by  $B_\nu = B_\lambda \lambda^2$ . The  $\lambda$  of the maximum in  $B_\nu$  will clearly be at a greater wavelength than for  $B_\lambda$  due to the  $\lambda^2$  weighting.

d) For the 250 K temperature use the graph to find the wavenumbers and wavelengths at which the Planck functions fall to about 10% of the maximum. This is an estimate of the wavelength range of significant emission.

Reading off the graph I get 5 to 40  $\mu\text{m}$  for  $B_\lambda$  and 50 to 1450  $\text{cm}^{-1}$  for  $B_\nu$ .

3. The integral of Planck function across a spectral band is part of a calculation of broadband fluxes and heating rates. The IDL file has a section `IntegratePlanck` that performs numerical integration of the Planck function between two wavenumbers. Examine the code to see how the trapezoidal integration is implemented. Perform a numerical integration across the whole longwave spectrum for blackbodies with temperatures of 250 and 300 K.

a) Compare the broadband results with fluxes from Stefan-Boltzmann Law.

Temperature	$\int B_\nu d\nu$ ( $\text{W m}^{-2} \text{ sr}^{-1}$ )	$\sigma T^4$ ( $\text{W m}^{-2}$ )	$\sigma T^4 / \pi$ ( $\text{W m}^{-2} \text{ sr}^{-1}$ )
300	146.20	459.27	146.19
250	70.50	221.48	70.50

The Stefan-Boltzmann Law is for flux, while the Planck function is for radiance, hence the factor of  $\pi$ , which comes from integrating the isotropic Planck function to get hemispheric flux.

b) Integrate the Planck functions at the two temperatures over the 8.5 to 12.5  $\mu\text{m}$  spectral window in the Earth's atmosphere. What fraction of the total blackbody radiation is emitted in the window for each temperature?

The spectral range in wavenumber is about 800 to 1175  $\text{cm}^{-1}$  ( $\nu = (10000 \text{ }\mu\text{m cm}^{-1})/\lambda$ ).

For 300 K,  $\int B_\nu d\nu = 38.12 \text{ W m}^{-2} \text{ sr}^{-1}$  or 26% of the broadband value.

For 250 K,  $\int B_\nu d\nu = 15.05 \text{ W m}^{-2} \text{ sr}^{-1}$  or 21% of the broadband value.

4. The Planck spectral integral may be calculated accurately and quickly with series expansions.

a) For the atmospheric window band in question 2, calculate  $x = c_2\nu/T$  to determine which series expansion to use. Finish the IDL code for the appropriate series expansion.

The minimum  $x$  is for 800  $\text{cm}^{-1}$  at 300 K or  $x = 3.84$ , so the high frequency expansion is appropriate. The band integration is done by evaluating the following series expansion at the lower and upper wavenumbers and subtracting. See the IDL solution code.

$$\int_\nu^\infty B_\nu(T) d\nu = c_1 \left(\frac{T}{c_2}\right)^4 \sum_{n=1}^{\infty} e^{-nx} \left[ \frac{x^3}{n} + \frac{3x^2}{n^2} + \frac{6x}{n^3} + \frac{6}{n^4} \right]$$

b) Compare the trapezoidal rule and series expansion values to test the expansion. How many terms do you need to get agreement to better than 0.1%.

Two terms ( $n=1,2$ ) gives 0.015% accuracy for  $T=300$  K and 0.003% for  $T=250$  K.