ATOC/ASTR 5560 Homework 6 Solutions
Due: December 12, 2001

1. *Compute the length of day and daily mean top-of-the-atmosphere solar insolation (W/m^2) at $65°$N on about July 21 for* **today** *and* **130,000 years ago**. *From Figure 6.2 in Liou (1992) the eccentricity then was about 0.038, the longitude of the perihelion from the vernal equinox was about $45°$, and obliquity was $24.0°$. Assume that the longitude of the earth and the true anomaly are linearly proportional to time. What is the relative size of the change in solar flux due to obliquity versus that due to the sun-earth distance?*

   The length of a day is $\frac{24 \text{ hrs}}{2\pi}(2H)$ where

$$\cos H = -\tan\phi \tan\delta \ ,$$

   is the hour angle at sunset and ϕ is the latitude and δ is the solar declination. The daily mean top of atmosphere solar insolation is

$$\bar{F} = \frac{S}{\pi}\left(\frac{r_0}{r}\right)^2 (H \sin\phi \sin\delta + \sin H \cos\phi \cos\delta)$$

   where $S = 1370$ W/m^2 is the solar constant, and r is the earth-sun distance. The solar declination is obtained from

$$\sin\delta = \sin\epsilon \sin(\nu + \omega) = \sin\epsilon \sin\lambda \ ,$$

   where ν is the true anomaly, and ω is the longitude of the perihelion. The earth sun distance is

$$r = \frac{a(1-e^2)}{a + e\cos\nu} \approx \frac{r_0(1-e^2)}{1 + e\cos\nu} \qquad \left(\frac{r_0}{r}\right)^2 \approx (1 + e\cos\nu)^2$$

   At perihelion $r_{min} = a(1-e)$ and at aphelion $r_{max} = a(1+e)$. The assumption that the true anomaly and earth longitude are linear in time means

$$\lambda = 2\pi\frac{t}{365} \qquad \nu = \lambda - \omega$$

   where t is the time in days from the vernal equinox. A month after northern summer solstice $\lambda = 2\pi/3 = 120°$.

   The current longitude of the perihelion is 13 days after northern winter solstice, so $\omega = 1.57\pi = 283°$. The current eccentricity is $e = 0.017$, and obliquity is $\epsilon = 23.5°$. Using these numbers for July 21 the solar declination is $\delta = 20.2°$ and the hour angle at sunset is $H = 2.48$ radians, so the current length of day is 18.9 hours. The approximate true anomaly is $\nu = 197°$, so $\left(\frac{r_0}{r}\right)^2 = 0.968$, and $\bar{F} = S\left(\frac{r_0}{r}\right)^2 0.3246 = 0.314S = 430$ W/m^2.

   For 130,000 years ago $e = 0.038$, $\epsilon = 24.0°$, and $\omega = 45°$ (perihelion on May 12). The calendar is defined by the seasons, so the vernal equinox is always March 21. The solar declination is $\delta = 20.6°$ and the hour angle at sunset is $H = 2.52$ radians, so the length of day was 19.2 hours. The approximate true anomaly is $\nu = 75°$, so $\left(\frac{r_0}{r}\right)^2 = 1.020$, and $\bar{F} = S\left(\frac{r_0}{r}\right)^2 0.3294 = 0.336S = 460$ W/m^2.

   The daily insolation at $65°$N on July 21 was 30 W/m^2 higher 130,000 years ago. A factor of 1.020/0.968=1.054 is due to the closer earth-sun distance, while a factor of 0.3294/0.3246=1.015 is due to the increased obliquity, so the earth-sun distance is significantly more important in this case.

2. *a) Calculate the radiative forcing for a solar constant increase of 0.1% at sunspot maximum. Compare this with the* current *radiative forcing from the anthropogenic increase in trace gases.*

The radiative forcing for a perturbation is obtained from the net flux

$$N = \frac{S}{4}(1-\bar{r}) - F_{LW} \ ,$$

where S is the solar constant, $\bar{r}$ is the mean global albedo, and F_{LW} is the mean outgoing longwave flux.

For a change in the solar constant it is the change in mean absorbed solar flux that matters, so

$$\Delta Q = \frac{\Delta S}{4}(1-\bar{r})$$

$$\Delta Q = \frac{0.001(1365 \text{ W m}^{-2})}{4}(1-0.30) \ = 0.24 \text{ W m}^{-2}$$

The radiative forcing is positive, so the Earth would warm.

The current radiative forcing from trace gases is about 2.5 W/m^2. The total solar irradiance changes over the sunspot cycle do not have significant climate effects.

b) Derive the climate sensitivity parameter G_0 for surface temperature with no climate feedbacks using a current surface temperature of $288°$K. It is easiest to assume that the outgoing longwave flux is proportional to the emitted surface longwave flux, $F_{LW} = (1-g)\sigma T_s^4$, where g is a normalized greenhouse factor. Why is g constant for no climate feedbacks?

The climate sensitivity for no feedbacks is a fundamental property of radiation physics. The same result is obtained using complex radiative convective equilibrium models or the simple assumption here. The outgoing longwave flux is equal to the absorbed solar flux

$$F_{LW} = (1-g)\sigma T_s^4 \ = \ \frac{S}{4}(1-\bar{r}) \ = \ \sigma T_e^4$$

The climate sensitivity with no feedbacks is the direct temperature response,

$$G_0 = \left(\frac{\partial F_{LW}}{\partial T_s}\right)^{-1} \ ,$$

which is how much the climate has to warm up or cool down to come back into equilibrium. The partial derivative is

$$\frac{\partial F_{LW}}{\partial T_s} = 4(1-g)\sigma T_s^3 \ = \frac{4F}{T_s}$$

Therefore the no feedback climate sensitivity is

$$G_0 = \frac{T_s}{4F_{LW}} \ = \frac{288\text{K}}{4(240 \text{ W m}^{-2})} = 0.30 \text{ K W}^{-1} \text{ m}^2$$

No climate feedbacks imply that atmospheric properties do not change in response to the radiative forcing, only the temperature changes. Since the atmosphere does not change, e.g. the water vapor is the same, the greenhouse effect is the same and g is constant. Note: to include feedbacks the greenhouse parameter would depend on temperature $g(T)$.

c) What is the resulting no feedback global surface temperature change for the current radiative forcing from anthropogenic trace gases?

The current radiative forcing from anthropogenic trace gases is about 2.5 W/m^2. The no feedback surface temperature change is therefore $\Delta T_s = G_0 \Delta Q = 0.75$ K.

3. *a) Suppose cloud cover of all types* decrease *by 5% (i.e. 0.03 for a cloud amount of 0.60) for each degree of surface temperature increase (and all other cloud properties stay constant). What is the feedback factor f_{cc} for this hypothetical cloud feedback? Use ERBE cloud radiative "forcing" results for the global mean net effect of clouds. Is this a positive or negative feedback?*

   The total (shortwave + longwave) cloud radiative forcing results from ERBE are -17 W/m^2. The negative "forcing" means that the shortwave reflection effect from clouds dominates the longwave greenhouse warming effect, so that clouds cool the Earth. Since cloud cover is hypothesized to decrease with increasing temperature, this is a positive feedback (warmer → less clouds → more shortwave absorbed).

   The definition of the feedback factor for the cloud fraction variable A_c is

$$f = G_0 \frac{\partial N}{\partial A_c} \frac{\partial A_c}{\partial T_s}$$

$$f_{cc} = (0.3 \text{ K W}^{-1} \text{ m}^2) \, \frac{-17 \text{ W/m}^2}{A_c} \frac{-0.05 A_c}{1 \text{ K}} = 0.255$$

   b) Climate models show that the water vapor feedback by itself increases the no feedback climate sensitivity by a factor of 1.6 (i.e. $G_{wv} = 1.6 G_0$). What is the total climate sensitivity including the water vapor feedback and the cloud feedback in part a?

   Using linear control theory the total climate sensitivity is obtained from summing the feedback factors:

$$G = \frac{G_0}{1 - \sum_j f_j}$$

   The water vapor feedback is obtained from the climate sensitivity ratio:

$$f_{wv} == 1 - \frac{G_0}{G} = 1 - 1/1.6 = 0.375$$

   Therefore the total feedback factor is $f_{tot} = f_{wv} + f_{cc} = 0.63$. The climate sensitivity with both feedbacks is

$$G = \frac{G_0}{1 - \sum_j f_j} \;=\; \frac{0.3 \text{ K W}^{-1} \text{ m}^2}{1 - 0.63} = 0.81 \text{ K W}^{-1} \text{ m}^2$$

   c) It is unlikely that the change in cloud cover for all cloud types would be the same. Discuss how changing the cloud cover of i) low altitude stratus clouds over the ocean and ii) thin high altitude cirrus clouds over land would cause radiative effects of opposite signs.

   The radiative effect of stratus clouds over the ocean is dominated by the shortwave effect. The temperature of boundary layer stratus clouds is almost as warm as the surface, so their longwave effect is very small. However, their optical depth is moderately high and they are over a dark surface, so they reflect much more sunlight back to space.

   The radiative effect of thin cirrus clouds over land is dominated by the longwave effect. Since these clouds are optically thin and primarily scatter sunlight forward, their albedo is small. However, cirrus clouds are very cold and have a substantial longwave emissivity (though often < 1) so they can greatly reduce the outgoing longwave radiation. They effectively block the IR window radiation, because the Planck function for upper tropospheric temperatures is much less than for surface temperatures.

   Therefore, increasing thin cirrus clouds would tend to have a warming effect, while increasing marine stratus clouds would have a cooling effect.

4. *a) Consider a more realistic version of the Eddington gray radiative equilibrium model. In this version there is a completely transparent atmospheric window occupying a fraction f (weighted by the black-body spectrum) of the longwave spectrum. Assume f does not depend on temperature. For Earth's*

atmosphere, $f \sim 0.3$. The longwave spectrum outside of the atmospheric window is gray. Assume there is no shortwave absorption. Derive the following expression for the surface temperature in this model

$$T_s^4 = T_e^4 \, \frac{1 + \frac{3}{4}\tau^*}{1 + \frac{3}{4} f \tau^*}$$

where T_e is the effective blackbody temperature of the planet, and τ^* is the longwave optical depth outside the window.

This derivation mostly follows the "Gray Radiative Equilibrium Model" handout. Let the radiative transfer equation apply to the spectrum outside of the window, so

$$\mu \frac{dI}{d\tau} = I - B \qquad I = \int_{\nu \notin \text{window}} I_\nu d\nu$$

Then the net flux is sum of the window and nonwindow parts:

$$F_{net} = f\sigma T_s^4 + 2\pi \int_{-1}^{+1} I(\mu)\mu d\mu$$

There is no downwelling flux in the window, and the upwelling flux is simply the surface emission $f\sigma T_s^4$. From the Eddington approximation, $I(\mu) = I_0 + I_1\mu$, the net flux is

$$F_{net} = f\sigma T_s^4 + \frac{4\pi}{3} I_1$$

From the radiative equilibrium assumption, $\frac{dF_{net}}{d\tau} = 0$, I_1 must be constant in τ since $f\sigma T_s^4$ does not depend on height.

Integrating the nonwindow radiative transfer equation by $d\mu$ gives

$$\frac{1}{3}\frac{dI_1}{d\tau} = I_0 - B$$

Since I_1 is constant, we again have that $I_0 = B$. Integrating the RTE by $\mu d\mu$ gives

$$\frac{dI_0}{d\tau} = I_1 \quad \text{or} \quad \frac{dB}{d\tau} = I_1$$

Therefore the Planck function is again linear in optical depth

$$B(\tau) = B(0) + I_1 \tau$$

We now apply boundary conditions at the top of the atmosphere. First, there is no downwelling longwave flux:

$$F^{\downarrow}(0) = 2\pi \int_{-1}^{0} I(\mu)\mu d\mu \; = \pi B(0) - \frac{2\pi}{3} I_1 \; = 0$$

From this we have $I_1 = \frac{3}{2}B(0)$. The second boundary condition at the top of the atmosphere is that the total upwelling longwave flux equals the absorbed solar flux F_{sun}:

$$F^{\uparrow}(0) = f\sigma T_s^4 + 2\pi \int_{0}^{+1} I(\mu)\mu d\mu \; = f\sigma T_s^4 + [\pi B(0) + \frac{2\pi}{3} I_1]$$

$$F_{sun} \; = \; f\sigma T_s^4 + 2\pi B(0) \qquad \text{TOA balance}$$

The surface equilibrium boundary condition is that the surface emission equals the sum of downwelling shortwave and longwave flux at the surface:

$$F_{sun} + F^{\downarrow}(\tau^*) \; = \; \sigma T_s^4$$

$$F^{\downarrow}(\tau^*) = \pi B(\tau^*) - \frac{2\pi}{3} I_1$$

Putting in the linear in optical depth expression for $B(\tau^*)$ and using $I_1 = \frac{3}{2}B(0)$, we find that the downwelling longwave flux at the surface is $F^{\downarrow}(\tau^*) = \frac{3}{2}\pi B(0)\tau^*$, so

$$F_{sun} + \frac{3}{2}\pi B(0)\tau^* \; = \; \sigma T_s^4 \qquad \text{surface balance}$$

Multiple the TOA balance equation by $\frac{3}{4}\tau^*$ and add to the surface balance equation to eliminate $B(0)$:

$$F_{sun}[1 + \frac{3}{4}\tau^*] = [1 + \frac{3}{4}\tau^* f]\sigma T_s^4$$

Solving for the surface temperature in terms of the effective temperature $F_{sun} = \sigma T_e^4$ gives

$$T_s^4 = \frac{1 + \frac{3}{4}\tau^*}{1 + \frac{3}{4}\tau^* f} T_e^4 \qquad\qquad \text{QED}$$

We can also eliminate T_s to solve for the constant $B(0)$ by multiplying the surface balance equation by f and subtracting from the TOA balance equation:

$$\pi B(0) = \frac{(1-f)F_{sun}}{2 + f\frac{3}{2}\tau^*}$$

The Planck function at the surface is then

$$\pi B(\tau^*) = \pi B(0) + \pi I_1 \tau^* \; = \pi B(0)[1 + \frac{3}{2}\tau^*] \; = (1-f)F_{sun}\frac{1 + \frac{3}{2}\tau^*}{2 + f\frac{3}{2}\tau^*}$$

We can solve for the air temperature at the surface using $(1-f)\sigma T^4(\tau^*) = \pi B(\tau^*)$ to get

$$T^4(\tau^*) = T_e^4 \frac{1 + \frac{3}{2}\tau^*}{2 + f\frac{3}{2}\tau^*}$$

b) Explain how having an atmospheric window prevents a runaway greenhouse effect in surface temperature as the longwave optical depth goes to infinity (as compared with the result for no window $f = 0$).

If there is an atmospheric window so $f > 0$, as the optical depth of the rest of the spectrum goes to infinity ($\tau^* \to \infty$) the surface temperature asymptotes to

$$T_s = T_e \left(\frac{1}{f}\right)^{1/4}$$

or $T_s = 345$ K for $f = 0.3$. This is very different from having no atmospheric window ($f = 0$) in which case the surface temperature increases without bounds

$$T_s = T_e \left(1 + \frac{3}{4}\tau^*\right)^{1/4}$$

Physically the window lets some of the surface emitted radiation escape to space (with no corresponding emission from the atmosphere downward), thereby cooling the surface. This also prevents the atmosphere from being as hot, limiting the contribution the atmosphere makes to the surface temperature.

c) What is the radiative equilibrium surface temperature for a longwave optical depth $\tau^* = 15$ and window parameter $f = 0.3$? Show that the surface/atmosphere temperature discontinuity, $T_s - T(\tau^*)$, is reduced substantially with the window as compared to the model with no window and optical depth τ^* adjusted to get the same surface temperature.

The surface radiative equilibrium temperature from part a is

$$T_s^4 = T_e^4 \, \frac{1 + \frac{3}{4}\tau^*}{1 + \frac{3}{4}f\tau^*}$$

For the effective planetary temperature of $T_e = 255$ K, $\tau^* = 15$, and $f = 0.3$ the surface temperature is $T_s = 329.9$ K.

In the derivation in part a we had that the air temperature at the surface is

$$T(\tau^*)^4 = T_e^4 \, \frac{\frac{1}{2} + \frac{3}{4}\tau^*}{1 + \frac{3}{4}f\tau^*}$$

For $\tau^* = 15$ and $f = 0.3$ the surface air temperature is $T(\tau^*) = 326.4$ K. The temperature discontinuity with the window is $\Delta T = 3.4$ K.

For the no window case we can solve for the equivalent gray optical depth

$$\tau^* = \frac{4}{3}\left[\left(\frac{T_s}{T_e}\right)^4 - 1\right]$$

For $T_s = 330$ K the gray optical depth is $\tau* = 2.4$. The surface air temperature with no window is

$$T(\tau^*) = T_e \left(\frac{1}{2} + \frac{3}{4}\tau^*\right)^{1/4}$$

which gives $T(\tau^*) = 314$ K or a discontinuity of $\Delta T = 15.8$ K.