ATOC/ASTR 5560 Homework 5 Solutions
Due: November 14, 2001

1. *a) Show that for an optically thin atmosphere and a very dark surface, the combined atmosphere-surface reflectivity is simply the sum of the reflectivity of the atmosphere and surface.*

   You can show this easily with the scalar adding formula for the two-stream method, though the same principle applies to the matrix version. The adding formula is

   $$R_t = R_a + \frac{T_a^2 R_s}{1 - R_a R_s}$$

   where R_a is the albedo of the atmosphere, T_a is the atmosphere transmittance, and R_s is the surface albedo. For an optically thin atmosphere $R_a \ll 1$ and $T_a \approx 1$, so $R_t \approx R_a + R_s/(1 + R_a R_s)$. If the surface is also dark, then $R_s \ll 1$ and $R_t \approx R_a + R_s$.

   b) In visible and near IR remote sensing the radiance is often normalized by the incident solar flux to give the reflectance *defined by $R = \pi I/(\mu_0 S_0)$. Calculate this reflectance in the direction $\theta = 30°, \phi = 140°$ from an atmosphere containing only a 1.0 km layer of the mineral aerosol from Lab 7. The surface is Lambertian with an albedo of 0.04. The direction to the sun is $\theta_0 = 30°, \phi_0 = 180°$. Calculate the total reflectance for two cases i) the actual aerosol Mie phase function, and ii) the Henyey-Greenstein phase function for the same asymmetry parameter.*

   The direction of the incident solar radiation is opposite the direction towards the sun, therefore $\theta' = 150°$, $\phi' = 0°$. The scattering angle is found from the incident and outgoing directions

   $$\cos\Theta = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi' - \phi)$$

   $$\cos\Theta = (0.866)(-0.866) + (0.5)(0.5)(-0.766) = -0.9415$$

   This gives a scattering angle of $\Theta = 160°$. Looking in the phase function file for the aerosol case, we see that the Mie phase function is $P_{Mie}(\Theta) = 0.3787$. The other optical properties in the Mie scattering file are extinction $\beta = 0.321\ \mathrm{km}^{-1}$, single scattering albedo, $\omega = 0.888$, and asymmetry parameter $g = 0.6815$. The Henyey-Greenstein phase function is defined by

   $$P_{HG}(\Theta) = \frac{1 - g^2}{(1 + g^2 - 2g\cos\Theta)^{3/2}}$$

   so $P_{HG}(\Theta) = 0.117$. A 1 km layer has an optical depth of $\tau = \beta\Delta z = 0.321$ (and the scaled optical depth is 0.13), so the layer is optically thin, and the first order scattering solution for radiance may be used:

   $$I_1^{\uparrow}(\mu, \phi) \; = S_0 \frac{\omega P(\Theta)}{4\pi}\,\frac{\tau}{\mu}$$

Converting this to reflectance gives

$$R_a(\mu, \phi) \; = \frac{\tau \omega P(\Theta)}{4\mu_0 \mu}$$

The reflectance from Lambertian surface is

$$R = \frac{\pi I}{\mu_0 S_0} = \frac{\pi}{\mu_0 S_0} \frac{R_s \mu_0 S_0}{\pi} = R_s$$

Then we just need to add the atmosphere and surface reflectance, $R_t = R_a + R_s$. Substituting in the numbers gives $R_{HG} = 0.011 + 0.04 = 0.051$ for the Henyey-Greenstein phase function and $R_{Mie} = 0.036 + 0.04 = 0.076$ for the Mie phase function. Note that the Henyey-Greenstein phase function is a very poor approximation for radiances, though it is addequate for fluxes.

2. *a) Calculate the delta-isotropic scaled optical properties (β, ω, g) for the $\lambda = 11$ μm cloud case in Lab 7.*

   The Mie results for the $\lambda = 11$ μm cloud case in Lab 7 are extinction $\beta = 38.87 \text{ km}^{-1}$, single scattering albedo $\omega = 0.4748$, and asymmetry parameter $g = 0.924$. The delta-isotropic scaling uses a scaling fraction of $f = g$, so that the scaled asymmetry parameter is $g' = 0$. The scaled optical properties are

$$\beta' = (1 - \omega f)\beta \; = [1 - (0.4748)(0.924)](38.87 \text{ km}^{-1}) = 21.8 \text{ km}^{-1}$$

$$\omega' = \frac{(1 - f)\omega}{1 - \omega f} \; = \frac{(1 - 0.924)(0.4748)}{1 - (0.4748)(0.924)} = 0.064$$

   b) Given the results in part a, what two approximate solutions to the thermally emitting and scattering radiative transfer equation would be both accurate and computationally efficient? Explain why. Hint: the second approximation is more accurate than the first.

   Water droplets in the mid-infrared are highly absorbing (the imaginary part of the index of refraction is near 0.1). The reason the single scattering albedo is near 0.5, rather than zero, is that the scattered radiation is in the forward scattering diffraction peak (still a useful concept even though this case is not in the geometric optics limit).

   Since the scaled single scattering albedo is so small, the nonscattering approximation should work well, using the scaled extinction $\beta' = 21.8 \text{ km}^{-1}$ to find the cloud emissivity. If more accuracy was desired, the first order or single scattering approximation for thermal radiative transfer would work very well. For thermal emission, the single scattering approximation involves an integral of the phase function times the emitted radiation over all angles.

3. *a) The similiarity parameter in radiative transfer is defined as $s = \sqrt{(1-\omega)/(1-\omega g)}$, where ω is the single scattering albedo and g is the asymmetry parameter. Show that the similiarity parameter is unchanged in delta scaling.*

The delta scaling relations are

$$\omega' = \frac{(1-f)\omega}{(1-f\omega)} \qquad g' = \frac{g-f}{1-f}$$

Substituting these into the similarity parameter definition gives

$$s' = \sqrt{\frac{1-(1-f)\omega/(1-f\omega)}{1-(1-f)\omega/(1-f\omega)(g-f)/(1-f)}}$$

$$s' = \sqrt{\frac{1-f\omega-(1-f)\omega}{1-f\omega-(g-f)\omega}} = \sqrt{\frac{1-\omega}{1-g\omega}} = s$$

b) Argue why the semi-infinite limit ($\tau \to \infty$) for the albedo of a homogeneous atmosphere should depend only on the similiarity parameter s and the sun angle μ_0, rather than on all three parameters, ω, g, and μ_0, separately.

The radiative properties are supposed to be invariant under delta scaling, for any value of f. Since the similiarity parameter is only a function of ω and g and is invariant under delta scaling, the albedo must depend only on s.

A proof goes as follows. Suppose the contrary, that the albedo R depends on both s and g, i.e. $R(s,g)$. Since $s(\omega,g)$ is invariant but g changes with delta scaling, $R(s,g)$ would also change with delta scaling. This contradicts the fact that $R(\omega,g)$ is invariant under delta scaling. Therefore, the albedo must depend only on the similarity parameter, i.e. $R(s)$.

c) By setting $g=0$, derive a simplified expression for the semi-infinite albedo in the Eddington approximation in terms of s and μ_0. Start with the Meador and Weavor semi-infinite solution (eq 22).

The Meador and Weavor semi-infinite solution for albedo is

$$R(\tau=\infty) = \frac{\omega(\alpha_2 + k\gamma_3)}{(1+k\mu_0)(k+\gamma_1)}$$

where $k = \sqrt{\gamma_1^2 - \gamma_2^2}$, $\alpha_2 = \gamma_1\gamma_3 + \gamma_2\gamma_4$, $\gamma_1 = 7/4 - \omega(1+3g/4)$, $\gamma_2 = -1/4 + \omega(1-3g/4)$, $\gamma_3 = 1/2 - 3g\mu_0/4$, and $\gamma_4 = 1 - \gamma_3$.

For $g = 0$, $\gamma_1 = 7/4 - \omega$, $\gamma_2 = -1/4 + \omega$, $\gamma_3 = 1/2$, and $\gamma_4 = 1/2$. Therefore, $\alpha_2 = 3/4$ and $k = \sqrt{3(1-\omega)}$. Substituting these values into the semi-infinite albedo formula gives

$$R(\tau=\infty, g=0) = \frac{\omega(3/4 + \sqrt{3(1-\omega)}/2)}{(1+\sqrt{3(1-\omega)}\mu_0)(7/4 - \omega - \sqrt{3(1-\omega)})}$$

The similarity parameter is $s = \sqrt{(1-\omega)/(1-\omega g)} = \sqrt{1-\omega}$, so $\omega = 1 - s^2$. Eliminating ω gives

$$R(\tau = \infty, s) = \frac{(1-s^2)(\frac{3}{4} + \frac{\sqrt{3}}{2}s)}{(1+\sqrt{3}s\mu_0)(\frac{3}{4} + \sqrt{3}s + s^2)} = \frac{\frac{\sqrt{3}}{2}(1-s^2)}{(1+\sqrt{3}s\mu_0)(s + \frac{\sqrt{3}}{2})}$$

d) Describe how the semi-infinite albedo R depends on ω and g using the expression for the similiarity parameter (a graph of R versus s might be useful here).

The semi-infinite albedo decreases smoothly as s goes from 0 to 1, decreasing more rapidly near $s = 0$ than near $s = 1$. The albedo decreases very rapidly from 1 as the coalbedo $(1 - \omega)$ leaves zero, due to multiple scattering amplifying the absorption. The albedo also decreases rapidly as the asymmetry parameter (g) approaches 1.

4. *This question is about the adding-doubling method in the two-stream approximation. The azimuthally symmetric discrete ordinate solar radiative transfer equation is obtained by using one quadrature angle μ_1 per hemisphere and two terms in the Legendre phase function series. For upwelling radiance I^+ the two-stream RTE is*

$$\mu_1 \frac{dI^+}{d\tau} = I^+ - \frac{\omega}{2}\left[(1+3g\mu_1^2)I^+ + (1-3g\mu_1^2)I^-\right] - \frac{\omega}{4\pi}(1-3g\mu_1\mu_0)S_0 e^{-\tau/\mu_0}$$

where $I^\pm = I(\tau, \pm\mu_1)$ are the discrete ordinate radiances. For downwelling radiance I^- the two-stream RTE is

$$\mu_1 \frac{dI^-}{d\tau} = -I^- + \frac{\omega}{2}\left[(1-3g\mu_1^2)I^+ + (1+3g\mu_1^2)I^-\right] + \frac{\omega}{4\pi}(1+3g\mu_1\mu_0)S_0 e^{-\tau/\mu_0}$$

a) Calculate the reflection R, transmission (T), and source ($S^\pm$) coefficients for an infinitesimally thin layer of optical depth $\delta\tau$. Note: for a homogeneous layer $R^+ = R^-$ and $T^+ = T^-$.

The initialization formulas may be obtained from the definition of the reflection, transmission, and source coefficients in the interaction principle:

$$I_0^+ = T^+ I_1^+ + R^+ I_0^- + S^+ \qquad I_1^- = T^- I_0^- + R^- I_1^+ + S^-$$

using the two-stream radiative transfer equation optically thin solution. For example, one can use a finite difference of the RTE

$$\mu_1 \frac{(I_0^+ - I_1^+)}{-\delta\tau} = I_1^+ - \frac{\omega}{2}\left[(1+3g\mu_1^2)I_1^+ + (1-3g\mu_1^2)I_0^-\right] - \frac{\omega}{4\pi}(1-3g\mu_1\mu_0)S_0 e^{-\tau/\mu_0}$$

and solve for I_0^+. Rearranging some this gives

$$I_0^+ = \left(1 - \frac{\delta\tau}{\mu_1}[1 - \frac{\omega}{2}(1+3g\mu_1^2)]\right) I_1^+ + \left(\frac{\delta\tau}{\mu_1}\frac{\omega}{2}(1-3g\mu_1^2)\right) I_0^- + \frac{\delta\tau}{\mu_1}\frac{\omega}{4\pi}(1-3g\mu_1\mu_0)S_0 e^{-\tau/\mu_0}$$

Looking at the interaction principle this results in the initialization formulas:

$$R = \delta\tau \frac{1}{\mu_1} \frac{\omega}{2} [1 - 3g\mu_1^2]$$

$$T = 1 - \delta\tau \frac{1}{\mu_1} [1 - \frac{\omega}{2}(1 + 3g\mu_1^2)]$$

$$S^{\pm} = \frac{\delta\tau}{\mu_1} \frac{\omega}{4\pi} (1 \mp 3g\mu_1\mu_0) S_0 e^{-\tau/\mu_0}$$

b) If we do the standard direct/diffuse solar radiation separation, what are the boundary conditions (i.e. incident radiances) on the layer in the interaction principle? Hence what terms in the interaction principle gives the radiance outgoing from the atmosphere?

With the direct/diffuse separation of solar radiation there is no incident diffuse radiation on the layer, i.e. I_0^- and I_1^+ are zero. Therefore, the interaction principle equation shows that the outgoing radiances are due to the source terms

$$I_0^+ = T^+ I_1^+ + R^+ I_0^- + S^+ = S^+ \qquad I_1^- = T^- I_0^- + R^- I_1^+ + S^- = S^-$$

c) Use the doubling relations to compute the albedo as a function of optical depth for $\omega = 1.0$ and $\omega = 0.99$. Use a solar zenith angle of 30° ($\mu_0 = 0.866$), asymmetry parameter $g = 0.85$, and a black surface. The double gaussian quadrature angle is $\mu_1 = 1/2$ with weight $w = 1$. Use delta scaling with $f = g^2$. Start with $\delta\tau = 1/1024$ and double to $\tau = 64$. Convert the discrete ordinate radiance to albedo and plot the two albedo curves. Hand in the program you use to do the computations.

First the input optical properties are delta scaled. Then the expressions in part (a) are used to initialize R, T, and $S^{\pm}$ before the doubling. The doubling relations are

$$R_{2n} = R + T\Gamma RT \qquad T_{2n} = T\Gamma T \qquad \Gamma = \frac{1}{1 - R^2}$$

$$S_{2n}^+ = S^+ + T\Gamma(\gamma^n S^+ + RS^-) \qquad S_{2n}^- = \gamma^n S^- + T\Gamma(S^- + RS^+\gamma^n)$$

where $\gamma = \exp(-\delta\tau/\mu_0)$ and n doubles at each step. To get from $\delta\tau = 1/1024$ to $\tau = 64$ requires 16 doubling steps.

The upwelling radiance at the one quadrature angle is first converted to flux with $F^{\uparrow} = 2\pi w\mu_1 I^+$, and then the flux is normalized by the incident flux $F^{\downarrow} = S_0\mu_0$ to get albedo. Therefore, $R = 2\pi w\mu_1 S^+/\mu_0$ where we use $S_0 = 1$ for the incident solar flux.

I used the following awk program. `dtau` is $\delta\tau$, `w` is ω, and `dtau1, w1, g1` are the delta scaled variables.

```
awk 'BEGIN {mu0=0.866; g=0.85; w=0.99; nd=16; dtau=1.0/1024; \
  mu1=0.5; pi=3.1415927;   f=g^2; \
  dtau1=dtau*(1-w*f); w1=(1-f)*w/(1-w*f); g1=(g-f)/(1-f);  \
  R=(dtau1/mu1)*0.5*w1*(1-3*g1*mu1^2); \
  T=1-(dtau1/mu1)*(1-0.5*w1*(1+3*g1*mu1^2)); \
  Sp=(dtau1/mu1)*w1*(1-3*g1*mu1*mu0)/(4*pi); \
  Sm=(dtau1/mu1)*w1*(1+3*g1*mu1*mu0)/(4*pi); \
  gam=exp(-dtau1/mu0); \
  for (i=1; i<=nd; i++) {n=2^(i-1); tau=dtau*2^i; \
     G=1/(1-R^2); R2=R+T*G*R*T; T2=T*G*T; \
     S2p=Sp+T*G*(gam^n*Sp+R*Sm); S2m=gam^n*Sm+T*G*(Sm+R*Sp*gam^n); \
     R=R2; T=T2; Sp=S2p; Sm=S2m;  \
     if (tau>0.1) printf "%2d %8.3f  %6.4f %6.4f  %6.4f %6.4f\n", \
        i, tau, R, T, 2*pi*mu1*Sp/mu0, 2*pi*mu1*Sm/mu0;  }  }'
```

The two albedos for the doubling steps with $\tau \geq 0.125$ are

τ	$R_{\omega=1.00}$	$R_{\omega=0.99}$
0.125	0.0082	0.0081
0.250	0.0165	0.0163
0.500	0.0337	0.0331
1.000	0.0690	0.0671
2.000	0.1397	0.1337
4.000	0.2681	0.2477
8.000	0.4551	0.3932
16.000	0.6513	0.5011
32.000	0.7998	0.5334
64.000	0.8921	0.5355