ATOC/ASTR 5560 Homework 4 Solutions
Due: October 29, 2001

1. *a) Derive an expression relating the Rayleigh scattering volume absorption coefficient for a particle size distribution to the integrated volume distribution (V, the volume of particles per volume of air).*

   The Rayleigh absorption cross section is

   $$C_{abs} = -\frac{8\pi^2 r^3}{\lambda} \text{Im}\left[\frac{m^2-1}{m^2+2}\right]$$

   To get the volume absorption coefficient, we need to integrate over the particle size distribution

   $$\beta = \int_0^\infty C_{abs} n(r) dr$$

   It is clear that the volume absorption coefficient will be proportional to the third moment of the size distribution, as is the particle volume fraction

   $$V = \int_0^\infty \frac{4\pi r^3}{3} n(r) dr$$

   Putting in the absorption cross section gives

   $$\beta = -\frac{6\pi}{\lambda} \text{Im}\left[\frac{m^2-1}{m^2+2}\right] \int_0^\infty \frac{4\pi r^3}{3} n(r) dr$$

   or in terms of the volume fraction

   $$\beta = -\frac{6\pi}{\lambda} \text{Im}\left[\frac{m^2-1}{m^2+2}\right] V$$

   b) What is the volume fraction for a log-normal size distribution with $N_{tot} = 100\ \text{cm}^{-3}$, $r_0 = 0.2\ \mu$m, and $\sigma = 0.40$? Calculate the volume absorption coefficient for ammonium sulfate at a wavelength of $10\ \mu$m ($m = 2.190 - 0.130i$). Compare with your results in lab 6.

   From the lecture notes a log-normal distribution has a particle volume fraction of

   $$V = \frac{4\pi}{3} N r_0^3 \exp(4.5\sigma^2)$$

   Using the above distribution parameters gives

   $$V = \frac{4\pi}{3}(100\ \text{cm}^{-3})(0.2\ \mu\text{m})^3 \exp[4.5(0.4^2)]\ = 6.88\ \mu\text{m}^3/\text{cm}^3 = 6.88 \times 10^{-12}$$

   So the aerosols are about 7 parts per trillion of air volume.

For the index of refraction of ammonium sulfate $m = 2.190 - 0.130i$, $\frac{m^2-1}{m^2+2} = 0.56 - 0.037i$. Putting the volume fraction in the Rayleigh formula derived above gives

$$\beta = \frac{6\pi}{10\ \mu\mathrm{m}}(0.037)(6.88 \times 10^{-12}) = 4.80 \times 10^{-13}\ \mu\mathrm{m}^{-1} = 4.80 \times 10^{-4}\ \mathrm{km}^{-1}$$

This agrees exactly with the numerical integration in the lab. Aerosols generally have negligible effects in the mid-infrared, unless they have larger mass contents, such as after a major volcanic eruption or in a dust storm.

2. *Rayleigh scattering for spherical particles is a limiting case of Mie scattering as the size parameter $x \to 0$. In this limit the scattered field wave coefficients a_n and b_n are all negligible except for a_1 which is*

$$a_1 = \frac{2i}{3}\frac{m^2-1}{m^2+2}x^3 ,$$

using the notation of Liou (1980) or Bohren and Huffman (1983).

Use the Mie theory results to derive the following quantities for Rayleigh scattering:

a) the scattering efficiency Q_{sca} and the extinction efficiency Q_{ext},

The scattering efficiency from Mie theory in terms of the a_n and b_n coefficients is:

$$Q_{sca} = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)(|a_n|^2 + |b_n|^2)$$

In the Rayleigh limit this reduces to

$$Q_{sca} = \left(\frac{2}{x^2}\right) 3|a_1|^2 = \frac{8}{3}x^4 \left|\frac{m^2-1}{m^2+2}\right|^2$$

The extinction efficiency from Mie theory is

$$Q_{ext} = \frac{2}{x^2}\sum_{n=1}^{\infty}(2n+1)\mathrm{Re}(a_n + b_n)$$

In the Rayleigh limit this reduces to

$$Q_{ext} = \frac{2}{x^2}3\mathrm{Re}(a_1) = \frac{6}{x^2}\mathrm{Re}\left(\frac{2i}{3}\frac{m^2-1}{m^2+2}x^3\right) = 4x\mathrm{Re}\left(i\frac{m^2-1}{m^2+2}\right) = -4x\mathrm{Im}\left(\frac{m^2-1}{m^2+2}\right)$$

b) the scattering amplitudes $S_1(\Theta)$ and $S_2(\Theta)$,

The Mie theory scattering amplitudes are

$$S_1(\Theta) = \sum_{n=1}^{\infty}\frac{2n+1}{n(n+1)}[a_n\pi_n(\cos\Theta) + b_n\tau_n(\cos\Theta)]$$

$$S_2(\Theta) = \sum_{n=1}^{\infty} \frac{2n+1}{n(n+1)} [b_n \pi_n(\cos\Theta) + a_n \tau_n(\cos\Theta)]$$

where the Mie angular functions are

$$\pi_n(\cos\Theta) = \frac{1}{\sin\Theta} P_n^1(\cos\Theta)$$

$$\tau_n(\cos\Theta) = \frac{d}{d\Theta} P_n^1(\cos\Theta)$$

For the Rayleigh limit the $n = 1$ angular functions are

$$P_1^1(\cos\Theta) = \sin\Theta$$

$$\pi_1(\cos\Theta) = \frac{1}{\sin\Theta} P_1^1(\cos\Theta) \ = 1$$

$$\tau_1(\cos\Theta) = \frac{d}{d\Theta} P_1^1(\cos\Theta) \ = \cos\Theta$$

The Rayleigh limit scattering amplitudes are

$$S_1(\Theta) = \frac{3}{2} a_1 \ = ix^3 \left(\frac{m^2 - 1}{m^2 + 2} \right)$$

$$S_2(\Theta) = \frac{3}{2} a_1 \cos\Theta \ = ix^3 \left(\frac{m^2 - 1}{m^2 + 2} \right) \cos\Theta$$

c) the phase function for intensity $P_{11}(\Theta)$.

The phase function for intensity is proportional to the square of the scattering amplitudes

$$P_{11}(\Theta) = \frac{2\pi}{k^2 \sigma_{sca}} \left(|S_1|^2 + |S_2|^2 \right) = \frac{2}{x^2 Q_{sca}} \left(|S_1|^2 + |S_2|^2 \right)$$

Putting in the Q_{sca} derived above and doing the algebra gves the Rayleigh phase function

$$P_{11}(\Theta) = \frac{3}{4} \left(1 + \cos^2\Theta \right)$$

d) Consider the case of a nonabsorbing sphere in this limit. What is the extinction efficiency derived above in this case? Is this result physical (compare with the scattering)? What might be the cause of this dilemma? In this limit of Mie theory what process does Q_{ext} really measure?

This limit of Mie theory has a certain inconsistency. This becomes apparent considering a nonabsorbing sphere, which implies that the index of refraction m has zero imaginary part. The Mie extinction efficiency in this limit is $Q_{ext} = 0$. This result is unphysical because the extinction efficiency should be equal to the scattering efficiency for no absorption. In this limit of Mie theory the extinction cross section is really the Rayleigh absorption cross section. If there is any absorption in the particle, in the Rayleigh limit $x \to 0$ the scattering cross section ($\propto x^6$) is negligible compared to the absorption cross section ($\propto x^3$), and so the extinction is equal to the absorption. The cause of the dilemma is the way we have taken the limit of the Mie theory extinction. If there is no absorption then the a_1 term of the extinction efficiency series is zero, so we need to go to higher terms to get the leading behavior (which would go as x^4).

3. *The parameters needed for a two-stream radiative transfer flux calculation in a scattering atmosphere are the optical depth τ, the single scattering albedo ω, and the asymmetry parameter g. Consider the cloud and haze layer from about 50 to 65 km in the Venusian atmosphere. The cloud and haze droplets are sulfuric acid. At a wavelength of 550 nm the index of refraction is $m = 1.45$. Say the optical depth of the cloud droplets is 25 and of the haze is 3. The asymmetry parameter of the cloud droplets is 0.78 and of the haze particles is 0.70.*

   a) Write down an expression for the total optical properties of the layer (τ, ω, and g) from these optical properties for the cloud and haze scattering.

   The optical properties that are proportional to the number of particles are the ones that can be added together, e.g. optical depth, volume scattering coefficient, etc. It does not work to add optical properties that are ratios, such as single scattering albedo and asymmetry parameter. Therefore the total optical properties are

$$\tau = \tau_c + \tau_h$$

$$\omega = \frac{\omega_c \tau_c + \omega_h \tau_h}{\tau}$$

$$g = \frac{g_c \omega_c \tau_c + g_h \omega_h \tau_h}{\omega \tau}$$

   b) Compute the total τ, ω, and g for the layer.

   Since the index of refraction of the cloud and haze is real, there is no absorption and the single scattering albedo is $\omega_c = \omega_h = 1$. Thus the total single scattering albedo is $\omega = 1$. The total optical depth is

$$\tau = \tau_c + \tau_h = 25 + 3 = 28$$

   The total asymmetry parameter is

$$g = \frac{(0.78)(1)(25) + (0.70)(1)(3)}{(1)(28)} = 0.771$$

4. *You see the full moon through a uniform thin cloud and notice a well defined disk of light around the moon. The outer part of the disk has a brownish-red tinge. The diameter of the disk is 10 times that of the moon. What is the disk called? Why is it colored? What is the approximate radius of particles in the cloud? Comment on the width of the particle size distribution. What type of cloud is this?*

   The disk around the moon is called the corona. The corona consists of a central aureole (or disk) of light around the moon (or sun) and perhaps one or more rings further out. The corona is only a few degrees in radius (not to be confused with the 22° halo from scattering by hexagonal ice crystals).

The corona is due to forward scattering of the light from the moon by the water droplets or ice crystals in the cloud. If the cloud is optically thin (say $\tau < 1$) then we need only to consider single scattering of light, which is described by the phase function. The size parameter, $x = 2\pi r/\lambda$, for cloud droplets at visible wavelengths is large (> 50) and so the scattering is in the geometric optics limit. This means that there will be a strong and narrow diffraction peak in the phase function centered on the forward scattering direction ($\Theta = 0$). The width of the diffraction peak is inversely related to the size parameter – nearly all the diffracted light is within $\Theta \approx 3/x$ (in radians) ($\Theta \approx 170^\circ/x$). Since most of the scattered light is in the diffraction peak, a disk of light with the angular width of the peak is visible around the moon.

The width of the diffraction peak (and radius of the rings) depends on the size parameter, and hence the wavelength of light. The longer visible wavelengths (red, 0.7 μm) will therefore have a wider corona disk than the shorter wavelengths (blue, 0.45 μm). The inner part of the corona disk may appear somewhat bluish, while the outer part has only the longer wavelengths, and hence will appear reddish.

A cloud contains a range of particle sizes, which smears out the corona. This blurring tends to muddy the colors some. But since the corona is well defined the size distribution must be relatively narrow. If the cloud had a wide range of droplet sizes then the corona would be blurred by the different widths of diffraction patterns from the droplets, but there would still be a central aureole. If the cloud optical depth is larger, then significant multiple scattering will tend to broaden and weaken the observed corona.

The scattering angle at the edge of the corona disk (Θ) is half the diameter of the corona, and the diameter of the moon is 0.5°. Therefore,

$$\Theta \approx \frac{170^\circ}{x} = (0.5^\circ)10/2 = 2.5^\circ$$

Solving for the size parameter gives

$$x \approx \frac{170^\circ}{2.5^\circ} = 68$$

The outer part of the corona disk is red so use wavelength of $\lambda = 0.65\ \mu$m to obtain the particle radius from the size parameter:

$$r = \frac{x\lambda}{2\pi} = \frac{68(6.5\ \mu\mathrm{m})}{2\pi} = 7.0\ \mu\mathrm{m}$$

With the small particle radius this is almost certainly a liquid cloud, probably altostratus since it is optically thin.