ATOC/ASTR 5560 Homework 3 Solutions

Due: October 10, 2001

1. *Use the approximation for the Ladenburg and Reiche function in the notes to calculate the equivalent width of the 183.3 GHz water vapor line for two layers. The first layer (0 to 1 km in a midlatitude summer atmosphere) has* $u = 1.15\,\mathrm{g/cm^2}$*, pressure* $p = 958\,\mathrm{mb}$*, and temperature* $T = 292\,\mathrm{K}$*. For the first layer the line halfwidth is* $\alpha = 0.0912\,\mathrm{cm^{-1}}$ *and line strength is* $S = 2.66\,\mathrm{cm/g}$*. The second layer (12 to 13 km in a midlatitude summer atmosphere) has* $u = 0.000392\,\mathrm{g/cm^2}$*, pressure* $p = 194\,\mathrm{mb}$*, and temperature* $T = 219\,\mathrm{K}$*. For the second layer the line strength is* $S = 4.29\,\mathrm{cm/g}$*. Calculate the line halfwidth, given that the halfwidth temperature coefficient is 0.64.*

   What curve of growth regime (limit) is each layer in? How close are the equivalent width formulas for these limits?

   The equivalent width of a single absorption line is

$$W = 2\pi\alpha L(x) \qquad x = \frac{Su}{2\pi\alpha}$$

   where α is the line halfwidth, S is the line strength, and u is the absorber amount. The Ladenburg and Reiche function can be approximated with a maximum error of 1% near $x = 1$ by

$$L(x) = x[1 + (\pi x/2)^{5/4}]^{-2/5}$$

   For the 0-1 km layer the parameter x is

$$x = \frac{(2.66\ \mathrm{cm/g})(1.15\ \mathrm{g/cm^2})}{2\pi(0.0912\ \mathrm{cm^{-1}}} = 5.34$$

   The Ladenburg and Reiche approximation and equivalent width are

$$L(x) = 1.794 \quad W = 1.028\ \mathrm{cm^{-1}}$$

   The actual Ladenburg and Reiche function is 1.814 or $W = 1.040\ \mathrm{cm^{-1}}$.

   For the 12-13 km layer we first have to find the absorption line halfwidth by scaling the halfwidth from the lower layer:

$$\alpha = \alpha_0 \left(\frac{p}{p_0}\right)\left(\frac{T_0}{T}\right)^n = (0.0912\ \mathrm{cm^{-1}})\left(\frac{194}{958}\right)\left(\frac{292}{219}\right)^{0.64} = 0.0222\ \mathrm{cm^{-1}}$$

   The x parameter is

$$x = \frac{(4.29\ \mathrm{cm/g})(0.000392\ \mathrm{g/cm^2})}{2\pi(0.0222\ \mathrm{cm^{-1}}} = 0.012$$

   and the Ladenburg and Reiche function and equivalent width are

$$L(x) = 0.012 \quad W = 0.00168\ \mathrm{cm^{-1}}$$

The equivalent width is much smaller due to the very low amount of water vapor in the upper troposphere.

The curve of growth regime can be determined from x parameter. The line center optical depth is $\tau_{cen} = 2x$. For the 0-1 km layer the center optical depth is greater than 10, and so the line is clearly saturated. This is the strong line limit. The equivalent width in this limit is proportional to $\sqrt{u}$:

$$W_{strong} = 2\sqrt{Su\alpha} \ = 2\sqrt{(2.66 \text{ cm/g})(1.15 \text{ g/cm}^2)(0.0912 \text{ cm}^{-1})} = 1.056 \text{ cm}^{-1}$$

which is close to the actual value above.

For the 12-13 km layer $x = 0.012$ so this is the weak line limit where the equivalent width is linear in the absorber amount:

$$W_{weak} = Su \ = (4.29 \text{ cm/g})(0.000392 \text{ g/cm}^2) = 0.00168 \text{ cm}^{-1}$$

2. *Show the sensitivity of band mean transmission to pressure by plotting the Goody random band model transmission as a function of pressure. Use the 400 to 500 cm^{-1} portion of the pure rotational water vapor band, for which the band model parameters at 260 K and 1013 mb are* $\bar{S}/\delta = 9.0 \text{ m}^2/\text{kg}$ *and* $\bar{S}/\bar{\alpha}\pi = 103 \text{ m}^2/\text{kg}$.

   a) Graph the Goody band model transmission as a function of pressure (log scale for p from 1 to 1000 mb) for water vapor absorber amount of $u = 20 \text{ kg/m}^2$ (2 cm). Assume the temperature is fixed at $T = 260$ K.

   The Goody random band model transmission is

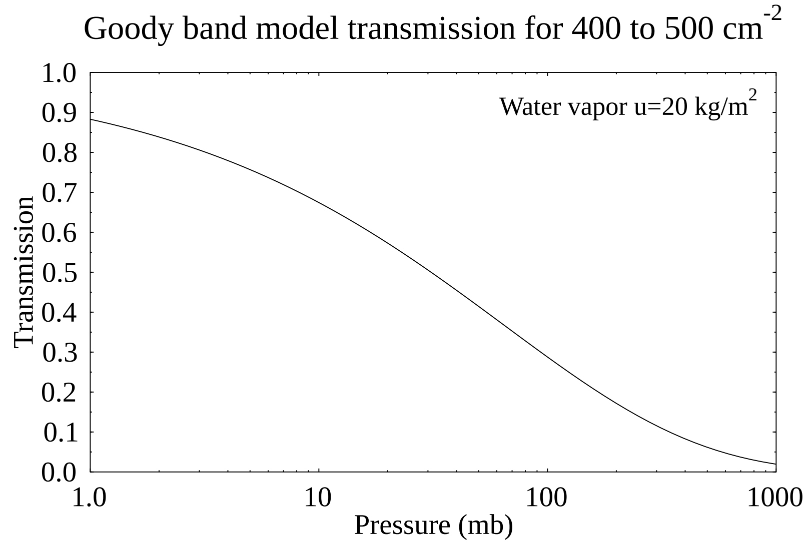
$$\bar{\mathcal{T}}(u) = \exp\left[-\frac{\bar{S}u}{\delta}\left(1 + \frac{\bar{S}u}{\pi\bar{\alpha}}\right)^{-1/2}\right]$$

   where $\bar{S}$ is the mean line strength and $\bar{\alpha}$ is the mean line width. The mean line width is proportional to pressure

$$\bar{\alpha} = \bar{\alpha}_0\left(\frac{p}{p_0}\right)$$

   where $\bar{\alpha}_0$ is the mean line width at the reference pressure $p_0 = 1013$ mb.

   The plot of band mean transmission vs. pressure shows that the transmission falls from 0.88 at 1 mb to 0.02 at 1000 mb.



b) Explain the change in transmission with pressure in terms of absorption line physics.

For fixed temperature the absorption line strengths do not change, and thus the mean optical depth is constant because the absorber amount is fixed. The effect of increasing pressure is to increase the width of the absorption lines. Many of the water vapor lines are strong so that the line cores remain saturated ($\tau_\nu \gg 1$) as the pressure is increased. Therefore, the effect of the increasing line width of these lines is to increase the fraction of the spectrum with essentially zero transmission. Another way of putting this is to say that the equivalent width of the lines increases with pressure in the strong line limit.

3. *Band mean transmission profiles from space to height z, $\mathcal{T}(\infty, z)$, and from the surface to z, $\mathcal{T}(0, z)$, for $\mu = 0.6$ have been calculated for bands from 700 to 750 cm^{-1} and from 1000 to 1050 cm^{-1}. These transmission profiles were calculated for the midlatitude summer standard atmosphere using MODTRAN3 and are available via anonymous ftp at* `ftp://nit.colorado.edu/pub/transprof3.dat`.

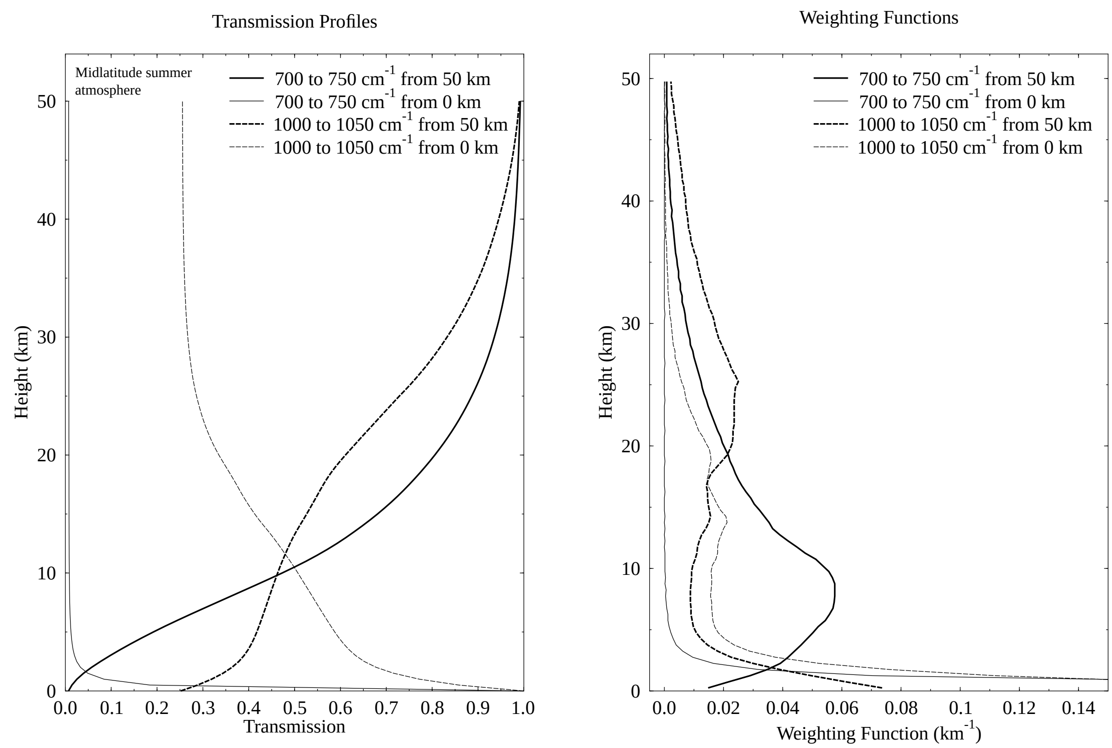
   a) Graph the four transmission profiles. Calculate and plot the weighting function referenced to space and referenced to the surface for both bands.

   The weighting functions are defined as

$$W_h(z) = \left| \frac{d\mathcal{T}(h, z)}{dz} \right|$$

   where h is the observer height and $\mathcal{T}(h, z)$ is the transmission profile from h to altitude z. The weighting function gives the contribution from the Planck function at each altitude to the to outgoing radiance. Numerically, the derivative of transmission is done with a centered finite difference:

$$W_h\left(\frac{z_i + z_{i+1}}{2}\right) = \left| \frac{\mathcal{T}(h, z_{i+1}) - \mathcal{T}(h, z_i)}{z_{i+1} - z_i} \right|$$



Why are the two weighting functions for the 700 to 750 cm^{-1} band so different?

The transmission always decreases away from the observer. However, from the ground, where the CO_2 density and aborption is very large, the transmission decreases very rapidly. Hence the weighting function peaks at the surface and falls off rapidly with height. The CO_2 density is very low in the upper atmosphere, so the transmission from space decreases slowly at first, and the weight function is small in the upper stratosphere. Then the transmission decreases more rapidly as more absorber is encountered, and the weighting function peaks. The weighting function decreases again for lower altitudes as the transmission becomes small.

b) Use the cooling to space approximation to compute and plot the cooling rate profiles in K/day for each band.
Why was $\mu = 0.6$ chosen for the angle to compute the transmission?

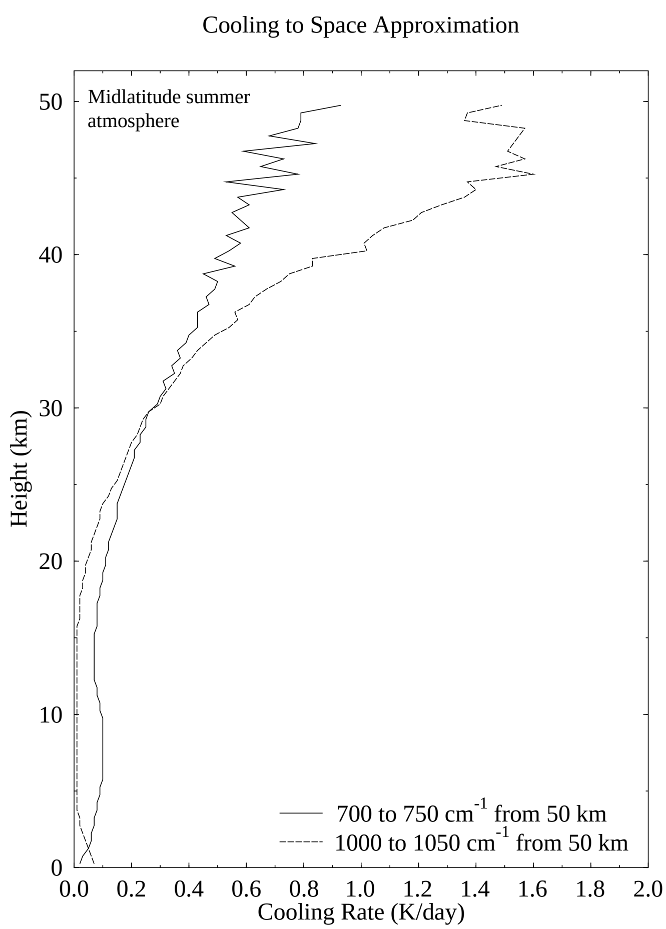
The cooling to space approximation gives the cooling rate profile as

$$\frac{dT}{dt} = \frac{1}{C_p \rho} \pi \int B_\nu[T(z)] \frac{d\mathcal{T}_\nu^f(z,\infty)}{dz} d\nu$$

where $B_\nu[T(z)]$ is the Planck function profile, $\mathcal{T}^f(z,\infty)$ is the flux transmission from height z to space, ρ is the air density, and C_p is the heat capacity of air. Since pressure of the levels is given in the file it is more convenient to use the pressure change across a layer rather than the height change and air density. We'll use the transmission at $\mu = 0.6$ to implement the diffusivity approximation for the *flux* transmission: $\mathcal{T}^f \approx \mathcal{T}(\bar{\mu})$. The integral of the Planck function may be done by evaluating the Planck function at the central wavenumber and multiplying by the width of the spectral band. The transmission function is already the band mean transmission. The cooling to space profile is then

$$\frac{dT}{dt} = 86400\,\mathrm{s/day}\frac{g}{C_p}\pi\ \Delta\nu\ B_{\nu_c}[(T_{i+1}+T_i)/2]\frac{d\mathcal{T}^f(z_{i+1},\infty) - d\mathcal{T}^f(z_i,\infty)}{p_i - p_{i+1}}$$

See the graph of the cooling rate profile. The "noise" on the upper part of the profiles is due to there being very small differences in the transmissions in the upper stratosphere and only having four decimal places of precision. The low density amplifies the cooling rate there.



What absorbing gases are causing the cooling rate features? For which band do you expect the cooling to space approximation to be more accurate? Why?

The stratospheric cooling in the 700 to 750 cm^{-1} band is mainly from CO_2 emission. The 700 to 750 cm^{-1} band also has some cooling to space from water vapor in the troposphere. The stratospheric cooling in the 1000 to 1050 cm^{-1} band is from ozone, and there is a minor amount of near surface cooling from water vapor.

The cooling to space approximation is more accurate for the CO_2 band than the ozone band. The ozone is mainly in the stratosphere, so the radiation emitted by the warm surface is absorbed by the ozone in the cold lower stratosphere. This exchange with the surface term is important for the ozone band. The CO_2 band is too optically thick in the troposphere to have exchange with the surface (except for just above the surface).

4. *This problem is about calculating the solar flux profile in a tropical atmosphere for 7700 to 14500 cm^{-1} (0.69 to 1.3 μm) using Fu and Liou's k-distribution. A file has been prepared with the water vapor density and k-distribution interpolated to the pressure and temperatures of each altitude level. The k-distribution weights are in the file. The file is available via anonymous ftp at* `ftp://nit.colorado.edu/pub/trp_kdist.dat`.

   a) Why do some of the k's decrease with height and some increase with height?

   Most of the k's have large weights Δg, and hence correspond either wavenumbers outside of water vapor bands or between the lines in water vapor bands. These k's decrease with height because decreasing pressure reduces the absorption in the wings of lines. The last k has the highest absorption and the least weight Δg. This k corresponds to the few wavenumbers near the line centers. These k values increase with height because the absorption of the line centers increases as the pressure is decreased.

   b) Calculate the band mean transmission and solar flux profile from 0 to 15 km in W/m^2 for solar angles of $\mu_0 = 1.0$ and $\mu_0 = 0.5$. Assume there is no reflection from the surface. For reference, the band transmission to the surface computed by MODTRAN3 for $\mu_0 = 1$ is 0.8393 for water vapor only and 0.8066 for all species.

   The band mean transmission is

$$\mathcal{T}_{\Delta\nu}(z_n) = \sum_{j=1}^{8} \Delta g_j \exp\left[-\frac{1}{\mu_0}\sum_{l=1}^{n} k_{j,l} u_l\right]$$

   where $k_{j,l}$ is the k-distribution mass absorption coefficient for layer l the j'th k, u_l is the absorber amount in layer l, and the l sum is over the n layer from the top down to level z_n. Since k-distribution values are provided at the layer boundaries, they should be averaged to obtain the layer values. The absorber amount can be obtained from the water vapor density by averaging: $u_l = \Delta z(\rho_{v,l} + \rho_{v,l+1})/2$. The awk program below calculates the band mean transmission profile. The idea is to start at the top and accumulate the optical depth τ_j for each k, using the layer mean absorption coefficient k_j and the absorber amount in each layer. Then for each level the band mean transmission from the top to the level is found by performing the weighted sum over the k's of Beer's law.

```
BEGIN {w[1]=0.71; w[2]=0.11;  w[3]=0.06;   w[4]=0.06;
       w[5]=0.04; w[6]=0.016; w[7]=0.0034; w[8]=0.0006;
       print "  F0=",F0,"W/m^2   mu0=",mu0;
       print "  Z   Trans  Fdown";}
{if ($1<15)
   { z2=$1; rho2=$4; u=(z1-z2)*(rho1+rho2)/2;
     Trans=0;
```

```
      for (j=1; j<=8; j++)
        { k2[j]=$(j+4);
          k=(k1[j]+k2[j])/2;
          tau[j]=tau[j]+k*u;
          Trans=Trans+w[j]*exp(-tau[j]/mu0)
        }
      Fdn = Trans*F0*mu0;
      printf "%5.1f %6.4f %6.2f\n",z1,Trans,Fdn;
    }
    z1=$1; rho1=$4; for (j=1; j<=8; j++) k1[j]=$(j+4);
}

tail -16 trp_kdist.dat | awk -f kdist.awk  -v mu0=1.0  -v F0=484.3
```

The solar flux results are in the following table.

Tropical Atmosphere Solar Flux Profile for 7700 to 14500 cm^{-1}.

	$\mu_0 = 1.0$		$\mu_0 = 0.5$	
Height (km)	$\mathcal{T}_{\Delta\nu}$	$F^{\downarrow}$ (W/m^2)	$\mathcal{T}_{\Delta\nu}$	$F^{\downarrow}$ (W/m^2)
14.0	1.0000	484.28	0.9999	242.13
13.0	0.9999	484.25	0.9998	242.10
12.0	0.9997	484.17	0.9995	242.03
11.0	0.9993	483.96	0.9988	241.85
10.0	0.9983	483.50	0.9972	241.48
9.0	0.9965	482.60	0.9942	240.74
8.0	0.9932	481.02	0.9892	239.53
7.0	0.9883	478.65	0.9816	237.70
6.0	0.9811	475.14	0.9704	234.99
5.0	0.9704	469.96	0.9552	231.30
4.0	0.9558	462.89	0.9348	226.37
3.0	0.9362	453.40	0.9075	219.75
2.0	0.9080	439.75	0.8708	210.86
1.0	0.8753	423.91	0.8302	201.04
0.0	0.8408	407.22	0.7883	190.89

The k-distribution value for the transmission at $\mu_0 = 1$ is very close to the value calculated by MODTRAN for water vapor. However, MODTRAN shows that there are other sources of extinction in this band, namely oxygen from the 0.76 μm band and molecular Rayleigh scattering.

c) Calculate the solar heating rate (K/day) in this band for these two sun angles.

The heating rate is calculated from the net flux convergence, and since the level pressures are given it is convenient to use this form

$$\left.\frac{dT}{dt}\right|_{rad} = \frac{g}{C_p}\frac{dF_{net}}{dp}$$

For the case of solar downwelling flux only the heating rate of a layer is

$$\left.\frac{dT}{dt}\right|_{rad} = \frac{g}{C_p}\frac{F_l^{\downarrow} - F_{l+1}^{\downarrow}}{p_{l+1} - p_l}$$

where $F_l^{\downarrow}$ is the incident flux at the top of the layer and $F_{l+1}^{\downarrow}$ is the exitting flux at the bottom of the layer.

The solar heating rate results are in the following table.

Tropical Atmosphere Solar Heating Rate Profile for 7700 to 14500 cm^{-1}.

Height (km)	$\mu_0 = 1.0$ dT/dt (K/day)	$\mu_0 = 0.5$ dT/dt (K/day)
14.5	0.01	0.01
13.5	0.01	0.01
12.5	0.02	0.02
11.5	0.05	0.04
10.5	0.10	0.08
9.5	0.18	0.14
8.5	0.27	0.21
7.5	0.37	0.28
6.5	0.50	0.38
5.5	0.65	0.47
4.5	0.81	0.56
3.5	0.98	0.68
2.5	1.28	0.83
1.5	1.35	0.84
0.5	1.29	0.79